Dynamic Asset Allocation with Loss Aversion and Illiquid Asset in a DC Pension Plan

Zhongfei Li

Joint work with Zheng Chen and Yan Zeng

Sun Yat-sen Business School, Sun Yat-sen University
Center for Financial Engineering and Risk Management, Sun Yat-sen University

The Fifth Asian Quantitative Finance Conference
Seoul, Korea
24-26 April, 2017

Outline

Background

- **Pension fund** management is attracting much attention due to population ageing.
- **Two main pension schemes:** Defined-benefit (DB) plan and defined-contribution (DC) plan.
  - **DB plan**: the benefits are defined in advance by the sponsor.
  - **DC plan**: only contributions are fixed and benefits depend solely on the investment returns of the fund.
- **Worldwide phenomenon:** DB pension plans are transformed to DC pension plans. DC pension plans have the advantage to ease the pressure of the social security system by transferring investment risk and longevity risk from sponsors to members.

Background (Cont'd)

- **Loss aversion** is considered in many related studies to explain the behavior of financial investors, such as Barberis et al. (2001), Berkelaar et al. (2004), Blake et al. (2013) and Curatola (2015).
- **Long investment horizon** makes that **inflation risk** should be incorporated in a DC pension plan investment model.
- **Illiquid assets** (e.g, fixed-term bank deposits, private equity investments and housing) constitute a large part of investment tools for pension funds.
- Ang et al. (2014) show that individuals typically hold more than 80% of their wealth in illiquid positions, and some types of the illiquid assets have investment horizons more than 10 years.
- **Pension funds** increase their holding in illiquid assets from 5% in 1995 to **20%** in 2010 (Global Pension Asset Study).

Literature

- Some researchers have studied the optimal asset allocation rules for DC pension plans (Boulier et al., 2001; Cairns et al., 2006; Emms, 2012; Blake et al., 2013; Blake et al., 2014; Konicz andMulvey, 2015; Chen and Delong, 2015).
- Loss aversion is introduced into the management of DC pension plans in some recent works (Blake et al., 2013; Guan and Liang, 2016; Chen et al., 2017).
- Desmettre and Seifried (2016) study a **static position** in **fixed-term securities** with a traditional dynamic trading strategy in stocks.

The work of this talk

- We investigate an optimal investment problem for a DC pension plan member who owns **loss aversion** and has a **stochastic salary flow**.
- The member can invest her wealth in a liquid market consisting of **an indexed bond, a stock** and **a risk-free bond**, and she has an opportunity to invest in **an illiquid asset**.
- Considering **inflation risk**, our aim is to maximize the expectation of the **S-shaped utility** of the **real terminal total wealth** at her retirement.
- We use a finite-horizon and continuous-time model.

Important assumptions

Comparing the literature, we make three important assumptions:

- The illiquid asset can only be traded at time 0 and provide a payment at the end of the investment (see Ílhan et al., 2005; Longstaff, 2009; Desmettre and Seifried, 2016).
- There is a salary risk.
- We adopt the S-shaped utility to describe the member's preference.

DC pension plan member

- Consider a DC pension plan member who participates in the labor market and makes continuously contribution to the pension plan before retirement at time T.
- The contribution rate is assumed to be a fixed percentage c of her salary.
- The member's **salary** follows

$$\frac{dY(t)}{Y(t)} = \mu_Y dt + \sigma_Y(\rho_{IY} dW_1(t) + \sqrt{1-\rho_{IY}^2} dW_2(t))$$
$$= \mu_Y dt + \Sigma_D' d\mathbb{W}(t)$$

  where $\Sigma_D = (\sigma_Y \rho_{IY}, \sigma_Y \sqrt{1-\rho_{IY}^2})'$, and $\mathbb{W}(t) := (W_1(t), W_2(t))'$ is a two-dimensional Brownian motion.
- It is reasonable to allow the salary to be correlated with **both the inflation risk and the stock market risk** introduced below.

Liquid market

- To capture the inflation risk, we introduce the **price level**

$$\frac{dP(t)}{P(t)} = idt + \sigma_I dW_1(t) = idt + \Sigma_I d\mathbb{W}(t),$$

where $\Sigma_I = (\sigma_I, 0)'$.

- The member considers a liquid financial market consisting of three tradeable assets:

Indexed bond

$$\frac{dI(t)}{I(t)} = rdt + \frac{dP(t)}{P(t)}$$

Stock

$$\frac{dS(t)}{S(t)} = \mu_S dt + \sigma_S(\rho_{IS} dW_1(t) + \sqrt{1-\rho_{IS}^2} dW_2(t))$$

Risk-free bond

$$\frac{dB(t)}{B(t)} = Rdt$$

Liquid market (Cont'd)

- Denote the volatility matrix

$$\Sigma = \begin{bmatrix} \sigma_I & 0 \\ \sigma_S \rho_{IS} & \sigma_S \sqrt{1-\rho_{IS}^2} \end{bmatrix},$$

- Unique market price of risk

$$\Xi = \Sigma^{-1} \begin{pmatrix} r+i-R \\ \mu_S - R \end{pmatrix} \equiv \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \frac{r+i-R}{\sigma_I} \\ \frac{1}{\sqrt{1-\rho_{IS}^2}}\left(\frac{\mu_S-R}{\sigma_S} - \frac{\rho_{IS}(r+i-R)}{\sigma_I}\right) \end{pmatrix}.$$

- stochastic discount factor (pricing kernel)

$$\frac{dH(t)}{H(t)} = -Rdt - \Xi' d\mathbb{W}(t), \; H(0) = 1.$$

Illiquid asset trading

The member also considers to invest an **illiquid asset** which

- offers a stochastic payoff $F(T)$ at time T,
- has a price of $F(0)$ at time 0.

Thus, if the member decides to **buy λ positions** of the illiquid asset at time 0, she will receive a lump-sum payment $\lambda F(T)$ at time T.

To preclude arbitrage opportunities, we **further assume** that short position in the illiquid asset and borrowing money to buy the illiquid asset are prohibited.

Investment strategy

- The pension account is endowed with **initial total wealth x_0**.
- At time 0, the member buys λ **positions** of the illiquid asset.
- Then the **initial liquid wealth** is $X^{\Pi,\lambda}(0) = x_0 - \lambda F(0)$.
- Let $\Pi(t) = (\pi_1(t), \pi_2(t))'$, where $\pi_1(t)$, $\pi_2(t)$ are the proportions of **liquid wealth** invested in the indexed bond and the stock, respectively; then $\pi_0(t) := 1 - \pi_1(t) - \pi_2(t)$ be the proportion of liquid wealth invested in the risk-free bond.
- The **liquid wealth process $X^{\Pi,\lambda}(t)$** follows

$$dX^{\Pi,\lambda}(t) = X^{\Pi,\lambda}(t)[Rdt + \Pi'(t)\Sigma(\Xi dt + d\mathbb{W}(t))] + cY(t)dt.$$

- The member's **terminal total wealth** is $X_T = X^{\Pi,\lambda}(T) + \lambda F(T)$.

Admissible strategy

- $\{(\Pi(t),\lambda) : t \in [0,T]\}$ is said to be **admissible** if
$$X^{\Pi,\lambda}(t) + cE_t[\int_t^T \frac{H(s)}{H(t)} Y(s)ds] \geq 0.$$
We denote the set of $\{(\Pi(t),\lambda)\}$ admissible portfolio strategies by $\mathcal{A}$.
It requires that the **liquid surplus of the member (her current liquid wealth plus the expected present value of future contribution)** is no less than 0 at each time.
- From the definition, let $t=0$, we have $0 \leq \lambda \leq \frac{x_0 + cE[\int_0^T \frac{H(s)}{H(t)} Y(s)ds]}{F(0)}$, showing that short position in the illiquid asset and borrowing money to buy the illiquid asset are prohibited.

S-shaped utility

The member is assumed to be **loss averse** with utility function

$$U(X_T) = \begin{cases} -A(\theta - X_T)^{\gamma_1}, & \text{if} \quad X_T \leq \theta, \\ B(X_T - \theta)^{\gamma_2}, & \text{if} \quad X_T > \theta. \end{cases}$$

where $A > B > 0$ captures loss aversion, $0 < \gamma_1, \gamma_2 < 1$ shows the S shape, and $\theta > 0$ is the reference point.

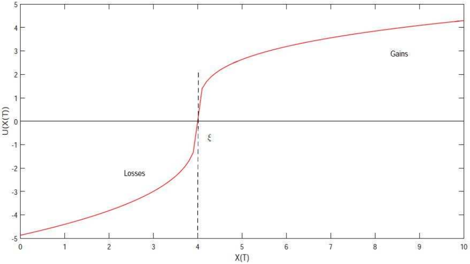

Figure 1: S-shaped utility.

Optimization problem

Now, the asset allocation problem can be formulated as the following optimization problem:

$$\max_{(\Pi,\lambda)\in\mathcal{A}(x_0,y_0)} \quad E[U(\underbrace{\frac{X^{\Pi,\lambda}(T)+\lambda F(T)}{P(T)}}_{real\ total\ wealth\ \frac{X_T}{P(T)}})]$$

$$\begin{aligned}
\text{s.t.} \quad & dX^{\Pi,\lambda}(t) = X^{\Pi,\lambda}(t)[Rdt + \Pi'(t)\Sigma(\Xi dt + d\mathbb{W}(t))] + cY(t)dt, \\
& dY(t) = Y(t)(\mu_Y dt + \Sigma_D' d\mathbb{W}(t)), \\
& dP(t) = P(t)(idt + \Sigma_I d\mathbb{W}(t)), \\
& X^{\Pi,\lambda}(0) = x_0 - \lambda F(0)
\end{aligned}$$

Solution method

We use the martingale method.

- Get rid of **contribution term** and make the market to be **self-financing**.
- For a **given** λ, we solve **terminal static optimization problem**.
- Look for an **optimal investment** λ^* in illiquid asset.
- Provide a characterization of the investor's **optimal investment strategy in the liquid market**.

Result without inflation

- We first consider a **baseline case to obtain explicit solutions**, and consider more general cases in the extension.
- In this case, inflation risk together with the indexed bond is not present.
- To obtain explicit solutions, we assume payment $F(T)$ is deterministic. (İlhan et al., 2005; Desmettre and Seifried, 2016)

Result without inflation (Cont'd)

Without inflation, our model is simplified into:

- The **risk-free bond**:
$$\frac{dB(t)}{B(t)} = Rdt$$
- The **stock**:
$$\frac{dS(t)}{S(t)} = \mu_S dt + \sigma_S dW_1(t)$$
- The **salary**:
$$\frac{dY(t)}{Y(t)} = \mu_Y dt + \sigma_Y dW_1(t)$$
- The **stochastic discount factor**: $H(t) = e^{-Rt-\frac{1}{2}\xi^2 t-\xi W_1(t)}$, **market price of risk**: $\xi = \frac{\mu_S - R}{\sigma_S}$
- Let $\pi(t)$ be the **proportion of the liquid wealth** invested in the stock, $1-\pi(t)$ in the risk-free bond.

Step 1

Denote the **expected present value of future contribution** by

$$D(t) = cE_t[\int_t^T \frac{H(s)}{H(t)} Y(s)ds].$$

By some straightforward calculation, we have

$$D(t) = \frac{1}{\beta}(e^{\beta(T-t)} - 1)cY(t), \qquad \forall t \in [0, T],$$

where $\beta = \mu_Y - R - \sigma_Y \xi$ and for short we denote $d_0 \equiv D(0)$.

Define a **liquid surplus process** $Z^{\pi,\lambda}(t) = X^{\pi,\lambda}(t) + D(t)$, which corresponds to the total present value of the future liquid wealth, comprising financial wealth and salary process.

Step 2

For a **fixed** investment λ in the illiquid asset, we solve

$$\max_{\pi} \quad E[U(Z^{\pi,\lambda}(T) + \lambda F(T))] = E[\bar{U}(\bar{Z}^{\pi}(T))]$$

where $\bar{U}(\bar{z}) \equiv U(\bar{z} + \lambda F(T))$ for $\bar{z} \in (0, +\infty)$, and the notation $\bar{Z}^{\pi}(t) \equiv Z^{\pi,\lambda}(t)$ stands for the liquid surplus process for a given λ. Similarly, we take $\bar{X}^{\pi}(t) \equiv X^{\pi,\lambda}(t)$ which represents the **liquid wealth process for a given λ**.

The problem is equivalent to the static problem

$$\begin{aligned} \max_{\bar{Z}^{\pi}(T)} \quad & E[\bar{U}(\bar{Z}^{\pi}(T))] \\ \text{s.t.} \quad & E[H(T)\bar{Z}^{\pi}(T)] = x_0 + d_0 - \lambda F(0), \\ & \bar{Z}^{\pi}(T) \geq 0 \end{aligned}$$

Step 2 (Cont'd)

Proposition 1

The optimal terminal liquid surplus (or wealth) for a fixed λ is

$$\bar{Z}^{\pi^*}(T) = \bar{X}^{\pi^*}(T) = \begin{cases} \left(\theta + \left(\frac{B\gamma_2}{y(\lambda)H(T)}\right)^{\frac{1}{1-\gamma_2}} - \lambda F(T)\right)^+, & \text{if} \quad H(T) < \bar{H}(\lambda), \\ 0, & \text{if} \quad H(T) \geq \bar{H}(\lambda), \end{cases}$$

where $\bar{H}(\lambda)$ satisfies $f(\bar{H}(\lambda)) = 0$ with

$$f(x) = \frac{1-\gamma_2}{\gamma_2}\left(\frac{1}{y(\lambda)x}\right)^{\frac{\gamma_2}{1-\gamma_2}}(B\gamma_2)^{\frac{1}{1-\gamma_2}} - \theta y(\lambda)x + A\theta^{\gamma_1}$$

and $y(\lambda) > 0$ is the corresponding Lagrange multiplier satisfying

$$E[H(T)\bar{Z}^{\pi^*}(T)] = x_0 + d_0 - \lambda F(0).$$

Remarks

- We find that the total wealth reaches above her reference point θ in good states, and it has a minimum guaranteed illiquid wealth $\lambda F(T)$ in bad states.
- When the reference point θ is chosen to be 0, the optimal terminal wealth is $\bar{X}^{\pi^*}(T) = (\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}} \vee \lambda F(T)$. (Desmettre and Seifried (2016))
- The illiquid asset trading helps the member to **lock in a minimum return as $\lambda F(T)$.**

Remark (Cont'd)

Investing in illiquid asset helps the member to hedge the financial risk efficiently and provide a **minimum guaranteed wealth of $\lambda F(T)$** even in bad states.

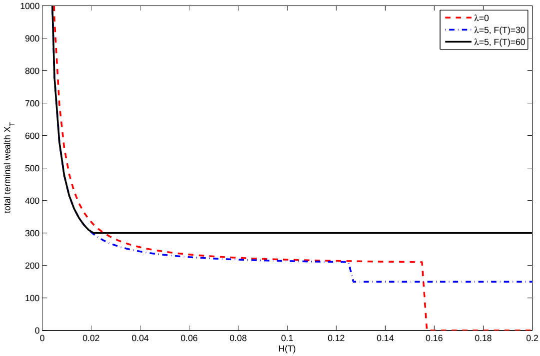

Figure 2: Effect of illiquid asset on the optimal terminal total wealth.

Step 3

- We further define a **new function** $V: \quad [0, \lambda_{max}] \to \mathbb{R}$,

$$V(\lambda) \equiv E[\bar{U}(\bar{Z}^{\pi^*}(T))] = E[U(X^{\pi^*,\lambda}(T) + \lambda F(T))]$$

where $\lambda_{max} \equiv \frac{x_0 + d_0}{F(0)}$.

- This function defines the expected utility of the terminal total wealth of the member for different illiquid trading strategies.
- We guarantee that the maximum of the function $V(\lambda)$ is attained.

Step 3 (Cont'd)

Proposition 2

The solution to the problem of $\max_{\lambda} V(\lambda)$ *exists and is unique, given by*

$$\lambda^* = \begin{cases} \underset{\lambda\in[0,\lambda_{max}]}{\arg\max}\ E\left[U_1\left(\left[\theta + \left(\frac{B\gamma_2}{y(\lambda)H(T)}\right)^{\frac{1}{1-\gamma_2}}\right] \vee \lambda F(T) - \theta\right)\right], & \text{if } H(T) < \bar{H}^*, \\ \lambda_{max}, & \text{if } H(T) \geq \bar{H}^*, \end{cases}$$

where function $U_1(\cdot)$ *is defined as* $U_1(X) = B \cdot (X)^{\gamma_2}$, *a concave function and* $y(\lambda)$ *is given as before.*

Remarks

- When $H(T) \geq \bar{H}^*$ a.e., the optimal illiquid asset investment strategy is putting the member's **total wealth** in the illiquid asset and at retirement she will get the total wealth as $\lambda_{max} F(T)$.
- When $H(T) < \bar{H}^*$, due to the fact that $\mathbb{P}(\theta + (\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}} = \lambda F(T)) > 0$, the function $V(\lambda)$ has a **non-trivial interior solution** in general. But it is hard to get the explicit form of λ^*, and we can only guarantee the **existence and uniqueness** of λ^*.

Remarks (Cont'd)

For optimal λ^*, the optimal terminal total wealth of the member is

$$X_T^* \quad = \quad X^{\pi^*,\lambda^*}(T) + \lambda^* F(T),$$

where $X^{\pi^*,\lambda^*}(T)$ is given by

$$X^{\pi^*,\lambda^*}(T) = \begin{cases} \Big(\theta + (\frac{B\gamma_2}{y^* H(T)})^{\frac{1}{1-\gamma_2}} - \lambda^* F(T)\Big)^+, & \textit{if} \quad H(T) < \bar{H}^*, \\ 0, & \textit{if} \quad H(T) \geq \bar{H}^*, \end{cases}$$

where $\bar{H}^* = \bar{H}(\lambda^*)$ and $y^* = y(\lambda^*)$ defined as before.

Step 4

Given λ^*, we derive the **closed-form expression** of $\pi^*(t)$.

Proposition 3

(i) The optimal liquid wealth of the member at time $0 \leq t < T$ is given by

$$X^{\pi^*,\lambda^*}(t) = \Big[\theta e^{-R(T-t)}N(d_1(\bar{H}^*)) + (\frac{B\gamma_2}{y^*H(t)})^{\frac{1}{1-\gamma_2}} e^{\Gamma(t)}N(d_2(\bar{H}^*)) - \lambda^* F(T)e^{-R(T-t)}N(d_1(\bar{H}^*))\Big]\mathbf{I}_{\{X^{\pi^*,\lambda^*}(T)>0\}} - D(t),$$

where

$$d_1(x) = \frac{log(\frac{x}{H(t)}) + (R - \frac{1}{2}\xi^2)(T-t)}{\xi\sqrt{T-t}},$$

$$d_2(x) = d_1(x) + \frac{\xi\sqrt{T-t}}{1-\gamma_2},$$

$$\Gamma(t) = \frac{\gamma_2}{1-\gamma_2}(R + \frac{1}{2}\xi^2)(T-t) + \frac{1}{2}(\frac{\gamma_2}{1-\gamma_2})^2\xi^2(T-t),$$

and $\bar{H}^*$ and y^* are given as before.

Step 4 (Cont'd)

Proposition 3 (Cont'd)

(ii) Let $\Lambda(t) = \frac{\theta e^{-R(T-t)}}{\xi\sqrt{T-t}}\Phi(d_1(\bar{H}^*)) + (\frac{B\gamma_2}{y^*H(t)})^{\frac{1}{1-\gamma_2}} e^{\Gamma(t)}(\frac{\Phi(d_2(\bar{H}^*))}{\xi\sqrt{T-t}} + \frac{N(d_2(\bar{H}^*))}{1-\gamma_2}) - \frac{\lambda^* F(T) e^{-R(T-t)}}{\xi\sqrt{T-t}}\Phi(d_1(\bar{H}^*)).$
Then, the percentage of wealth invested in the stock is

$$\pi^*(t) = \begin{cases} \underbrace{\frac{\xi/\sigma_S}{X^{\pi^*,\lambda^*}(t)}\Lambda(t)}_{Part\ 1} - \underbrace{\frac{\sigma_Y/\sigma_S}{X^{\pi^*,\lambda^*}(t)}D(t)}_{Part\ 2}, & if \quad \theta + (\frac{B\gamma_2}{y^*H(T)})^{\frac{1}{1-\gamma_2}} \geq \lambda^* F(T), \\ 0, & else, \end{cases}$$

where $N(\cdot)$ *denotes the cumulative standard normal distribution function and* $\Phi(\cdot)$ *denotes the standard normal density function.*

Remarks

- In the optimal investment strategy, part 1 illustrates her optimal investment strategy in liquid market in the presence of illiquid asset in absence of salary risk. It is adjusted by the **condition** $\theta + (\frac{B\gamma_2}{y^* H(T)})^{\frac{1}{1-\gamma_2}} \geq \lambda^* F(T)$**,** determined by the specific role of the illiquid asset in the model.
- Part 2 shows that the salary process exists in the portfolio and generates a **new hedging component regarding salary risk.**

Real liquid wealth process and real salary process

- In general case, to maximize the expected utility of **real wealth** $\frac{X^{\pi,\lambda}(T)+\lambda F(T)}{P(T)}$ at retirement T, we first denote $\hat{X}^{\Pi,\lambda}(t) := \frac{X^{\Pi,\lambda}(t)}{P(t)}$ and $\hat{Y}(t) := \frac{Y(t)}{P(t)}$.
- Using Itô's formula, we have the **real liquid wealth process** as

$$\begin{aligned} d\hat{X}^{\Pi,\lambda}(t) &= \hat{X}^{\Pi,\lambda}(t)[Rdt + \Pi'\Sigma(\Xi - \Sigma_I)dt - (i - \sigma_I^2)dt] \\ &\quad + \hat{X}^{\Pi,\lambda}(t)[\Pi'\Sigma - \Sigma_I{}']d\mathbb{W}(t) + c\hat{Y}(t)dt, \end{aligned}$$

and **real salary process** as

$$d\hat{Y}(t) = \hat{Y}(t)[(\mu_Y - i + \sigma_I^2 - \sigma_Y\rho_{IY}\sigma_I)dt + (\Sigma_D - \Sigma_I)'d\mathbb{W}(t)]$$

Real liquid wealth process and real salary process (Cont'd)

- Denote $\hat{\pi}_1(t) := \pi_0(t)$, $\hat{\pi}_2(t) := \pi_2(t)$, $\hat{\Sigma} := \begin{bmatrix} -\sigma_I & 0 \\ \sigma_S\rho_{IS} - \sigma_I & \sigma_S\sqrt{1-\rho_{IS}^2} \end{bmatrix}$ which is also nonsingular, and $\hat{\Xi} := \Xi - \Sigma_I$, $\hat{\Pi}(t) = (\hat{\pi}_1(t), \hat{\pi}_2(t))'$, after some calculations, we **rewrite the real liquid wealth process as**

$$d\hat{X}^{\hat{\Pi},\lambda}(t) = \hat{X}^{\hat{\Pi},\lambda}(t)[rdt + \hat{\Pi}'\hat{\Sigma}(\hat{\Xi}dt + d\mathbb{W}(t))] + c\hat{Y}(t)dt,$$

- The **new stochastic discount factor**

$$\frac{d\hat{H}(t)}{\hat{H}(t)} = -rdt - \hat{\Xi}'d\mathbb{W}(t).$$

Similar problem as before

- We formulate a similar optimization problem.
- By a series of simple but tedious calculations, the related process $\hat{X}^{\hat{\Pi},\lambda}(t) + \hat{D}(t)$ is self-financing, and $\hat{H}(t)[\hat{X}^{\hat{\Pi},\lambda}(t) + \hat{D}(t)]$ is a martingale.
- We adopt similar steps in former sections, and find that the inflation risk would have **significant** influence on the payment of the illiquid asset. It is reasonable to take a **conservative investment** in the illiquid asset compared with the baseline case.
- However, when payment $F(T)$ is stochastic in general, the closed-form solutions can not always be obtained.
- The extension is not completed and we will continue.

Conclusion

- A dynamic asset allocation problem for a **loss-averse** DC plan member with **salary risk** and **illiquid asset** is studied.
- We extend the framework of Desmettre and Seifried (2016) to DC pension plan management with **salary risk and S-shaped utility**.
- A **closed-form expression of optimal investment strategy** is derived by the martingale approach.
- Some interesting phenomena are found.

Future study

- Optimal retirement timing
- Incomplete market (uninsurable salary risk)
- Reference point adaptation
- More realistic constraint, e.g., $\text{contribution} = \max\{cY(t), M\}$

Thanks for your attention!