An Optimal Consumption of Necessity and Luxury, Investment, and Voluntary Retirement Choice Problem with Quadratic and HARA Utility

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Joint work with Prof. Hyeng Keun Koo (Ajou Univ.) and Prof. Kum-Hwan Roh (Hannam Univ.)

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Contents

Literature Reviews

- In asset pricing and portfolio selection, consider only a single consumption good, ignoring heterogeneity in goods ((Merton (1973, Econometrica), Mehra and Prescott (1985, JME), Campbell and Cochrane (1999, JPE))
- Recently consider a multi-good economy
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  - Wachter and Yogo (2010, RFS) - divide goods into two categories, *necessities* and *luxuries*

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Motivation

- People do not spend their wealth and time just for their consumption but use their resources to establish a good reputation, obtain esteem or solicit love and care from others.
- consumption of necessity → bliss level → Quadratic utility

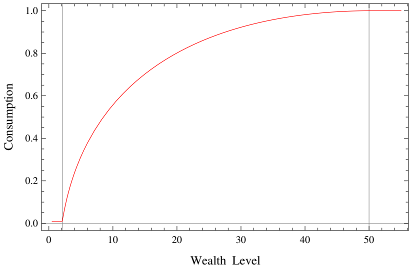

Figure 1: Consumption (Koo et al. (2016, SAA))

- luxury good
  - BMW, Jet airplane, Diamond etc
  - motive for bequest or giving etc

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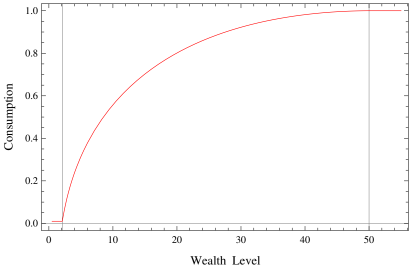

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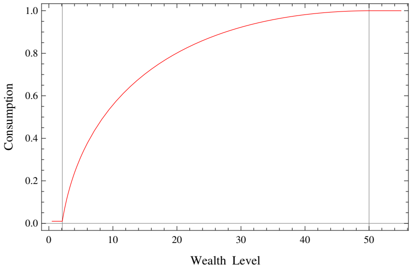

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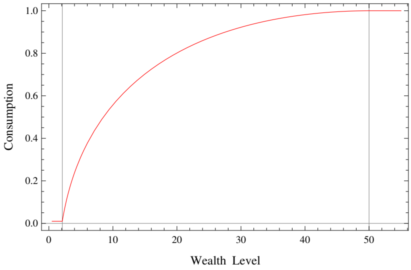

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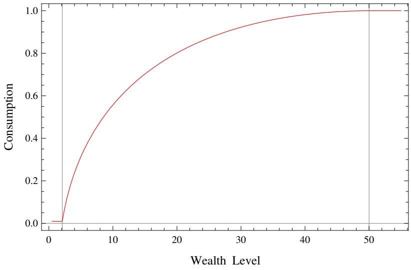

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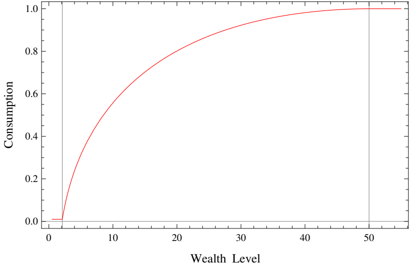

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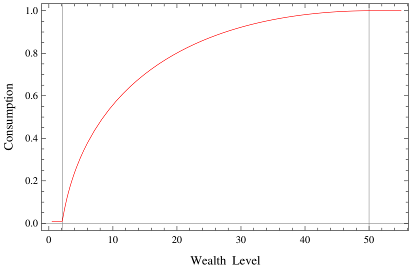

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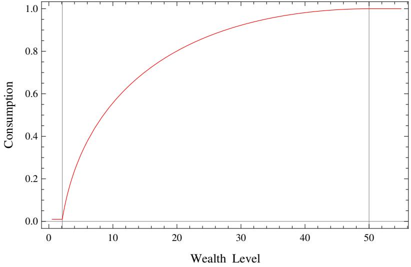

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The Financial Market

- $r > 0$: interest rate
- $dS_t/S_t = \mu dt + \sigma dB_t$: a risky asset
- $\theta := \frac{\mu - r}{\sigma}$: market price of risk
- $\zeta_t = \exp\{-rt\}$: discount process
- $Z_t := \exp\left\{-\theta B_t - \frac{1}{2}\theta^2 t\right\}$: exponential martingale
- $H_t := \zeta_t Z_t$: pricing kernel (or state price density)
- The equivalent martingale measure :
  $\tilde{\mathbb{P}}(A) := \mathbb{E}[Z_T \mathbf{1}_A], \quad \text{for } A \in \mathcal{F}_T$
- $\tilde{B}_t := B_t + \theta t$: a standard Brownian motion under the new measure $\tilde{\mathbb{P}}$ by Girsanov theorem

Wealth Process X_t

$$dX_t = \left[rX_t + \pi_t(\mu - r) - c_t - g_t + \epsilon \mathbf{1}_{\{0 \le t < \tau\}}\right] dt + \sigma \pi_t dB_t, \qquad (1)$$

with initial endowment $X_0 = x > -\epsilon/r$.

Main Optimization Problem

The agent's optimization problem is to maximize her/his expected utility

$$J(x; c, g, \pi, \tau) := \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left\{\alpha_1\left(c_t - Rc_t^2\right) + \alpha_2\frac{(g_t+L)^{1-\gamma}}{1-\gamma} - l\mathbf{1}_{\{0\le t<\tau\}}\right\}dt\right]$$

subject to the budget constraint

$$\mathbb{E}\left[\int_0^\tau H_t(c_t+g_t)dt + H_\tau\left(X_\tau + \frac{\epsilon}{r}\right)\right] \le x + \frac{\epsilon}{r}.$$

Here $\rho > 0$ is a subjective discount factor, $\alpha_i \ge 0$ $(i = 1, 2)$ with $\alpha_1 + \alpha_2 = 1$, $R > 0$, $L \ge 0$, $\gamma > 0$ $(\gamma \neq 1)$ is an agent's coefficient of relative risk aversion, and $l > 0$ is disutility from labor. Thus the value function of our optimization problem is given by

$$V(x) := \sup_{(c,g,\pi,\tau)\in\mathcal{A}(x)} J(x; c, g, \pi, \tau). \quad (2)$$

Benchmark Model without Labor Income

The agent wants to maximize her/his expected utility

$$J_1(x; c, g, \pi) := \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left\{\alpha_1\left(c_t - Rc_t^2\right) + \alpha_2 \frac{(g_t + L)^{1-\gamma}}{1-\gamma}\right\} dt\right]$$

subject to the budget constraint

$$\mathbb{E}\left[\int_0^\infty H_t(c_t + g_t) dt\right] \leq x.$$

$$U(x) := \sup_{(c,g,\pi)\in\mathcal{A}(x)} J_1(x; c, g, \pi). \tag{3}$$

For a Lagrange multiplier $\lambda > 0$, we obtain

$$\begin{aligned}
\widetilde{V}_1(\lambda) &:= \max_{(c,g,\pi)\in\mathcal{A}(x)} \left[\mathbb{E}\left[\int_0^\infty e^{-\rho t}\left\{\alpha_1(c_t - Rc_t^2) + \alpha_2 \frac{(g_t+L)^{1-\gamma}}{1-\gamma}\right\} dt\right] \right. \\
&\qquad \left. -\lambda \mathbb{E}\left[\int_0^\infty H_t(c_t+g_t)dt\right]\right] \\
&= \max_{(c,g,\pi)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left\{\alpha_1\left(c_t - Rc_t^2 - \frac{1}{\alpha_1}\lambda e^{\rho t} H_t c_t\right)\right.\right. \\
&\qquad \left.\left. +\alpha_2\left(\frac{(g_t+L)^{1-\gamma}}{1-\gamma} - \frac{1}{\alpha_2}\lambda e^{\rho t} H_t g_t\right)\right\} dt\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left\{\alpha_1 \cdot \widetilde{u}_1(\lambda e^{\rho t} H_t) + \alpha_2 \cdot \widetilde{u}_2(\lambda e^{\rho t} H_t)\right\} dt\right],
\end{aligned}$$

where

$$\widetilde{u}_1(y) = \max_{c\geq 0}\left\{c - Rc^2 - \frac{1}{\alpha_1} yc\right\} = \frac{\left(1 - \frac{1}{\alpha_1} y\right)^2}{4R}\mathbf{1}_{\{0<y\leq\alpha_1\}} + 0\cdot\mathbf{1}_{\{y>\alpha_1\}}$$

and

$$\begin{aligned}
\widetilde{u}_2(y) &= \max_{g\geq 0}\left\{\frac{(g+L)^{1-\gamma}}{1-\gamma} - \frac{1}{\alpha_2} yg\right\} = \left\{\frac{\gamma}{1-\gamma}\left(\frac{1}{\alpha_2}y\right)^{-\frac{1-\gamma}{\gamma}} + L\frac{1}{\alpha_2}y\right\}\mathbf{1}_{\{0<y\leq\alpha_2 L^{-\gamma}\}} \\
&\quad + \frac{L^{1-\gamma}}{1-\gamma}\mathbf{1}_{\{y>\alpha_2 L^{-\gamma}\}}
\end{aligned}$$

$$c_t^* = \begin{cases} \frac{1-\frac{1}{\alpha_1}y}{2R}, & \text{if } 0 < y \leq \alpha_1 \\ 0, & \text{if } y > \alpha_1, \end{cases}$$

$$g_t^* = \begin{cases} \left(\frac{1}{\alpha_2}y\right)^{-\frac{1}{\gamma}} - L, & \text{if } 0 < y \leq \alpha_2 L^{-\gamma} \\ 0, & \text{if } y > \alpha_2 L^{-\gamma}, \end{cases}$$

Remark

For later use, we define a quadratic equation as follows:

$$f(m) := \frac{1}{2}\theta^2 m^2 + \left(\rho - r - \frac{1}{2}\theta^2\right) m - \rho = 0. \qquad (4)$$

$f(m) = 0$ has two real roots $m_+ > 1$ and $m_- < 0$.

For $\alpha_1 > \alpha_2 L^{-\gamma}$,

$$\phi(y) = \mathbb{E}^{y_t=y}\left[\int_t^\infty e^{-\rho(s-t)}\left\{\left(\frac{\alpha_1\left(1-\frac{1}{\alpha_1}y_s\right)^2}{4R} + \frac{\alpha_2\gamma}{1-\gamma}\left(\frac{1}{\alpha_2}y_s\right)^{-\frac{1-\gamma}{\gamma}} + Ly_s\right)\right.\right.$$
$$\times \mathbf{1}_{\{0<y_s\le\alpha_2 L^{-\gamma}\}}$$
$$\left.\left.+\left(\frac{\alpha_1\left(1-\frac{1}{\alpha_1}y_s\right)^2}{4R} + \frac{\alpha_2 L^{1-\gamma}}{1-\gamma}\right)\mathbf{1}_{\{\alpha_2 L^{-\gamma}<y_s\le\alpha_1\}} + \frac{\alpha_2 L^{1-\gamma}}{1-\gamma}\mathbf{1}_{\{y_s>\alpha_1\}}\right\}ds\right].$$

By Feynman-Kac formula, we obtain the following ODE

$$\frac{1}{2}\theta^2 y^2\phi''(y) + (\rho-r)y\phi'(y) - \rho\phi(y)$$
$$+\left(\frac{\alpha_1\left(1-\frac{1}{\alpha_1}y\right)^2}{4R} + \frac{\alpha_2\gamma}{1-\gamma}\left(\frac{1}{\alpha_2}y\right)^{-\frac{1-\gamma}{\gamma}} + Ly\right)\mathbf{1}_{\{0<y\le\alpha_2 L^{-\gamma}\}}$$
$$+\left(\frac{\alpha_1\left(1-\frac{1}{\alpha_1}y\right)^2}{4R} + \frac{\alpha_2 L^{1-\gamma}}{1-\gamma}\right)\mathbf{1}_{\{\alpha_2 L^{-\gamma}<y\le\alpha_1\}} + \frac{\alpha_2 L^{1-\gamma}}{1-\gamma}\mathbf{1}_{\{y>\alpha_1\}} = 0.$$

Proposition

We derive

$$\phi(y) = \begin{cases} a_1 y^{m_+} + \frac{\gamma}{1-\gamma}\left(\frac{1}{\alpha_2}\right)^{-\frac{1}{\gamma}} \frac{1}{K} y^{-\frac{1-\gamma}{\gamma}} \\ \quad - \frac{1}{4\alpha_1(\rho-2r+\theta^2)R} y^2 + \frac{2LR-1}{2rR} y + \frac{\alpha_1}{4\rho R}, & \text{if } 0 < y \leq \alpha_2 L^{-\gamma} \\ b_1 y^{m_+} + b_2 y^{m_-} \\ \quad - \frac{1}{4\alpha_1(\rho-2r+\theta^2)R} y^2 - \frac{1}{2rR} y + \frac{\alpha_1}{4\rho R} + \frac{\alpha_2 L^{1-\gamma}}{\rho(1-\gamma)}, & \text{if } \alpha_2 L^{-\gamma} < y \leq \alpha_1 \\ c_2 y^{m_-} + \frac{\alpha_2 L^{1-\gamma}}{\rho(1-\gamma)}, & \text{if } y > \alpha_1, \end{cases}$$

$$b_1 = \frac{\frac{2-m_-}{4(\rho-2r+\theta^2)} + \frac{1-m_-}{2r} + \frac{m_-}{4\rho}}{(m_+ - m_-)R} \alpha_1^{1-m_+},$$

$$b_2 = \frac{\left(\frac{m_+\gamma}{1-\gamma} + 1\right)\frac{1}{K} + \frac{m_+ - 1}{r} - \frac{m_+}{\rho(1-\gamma)}}{m_+ - m_-} \alpha_2^{1-m_-} L^{1-\gamma+\gamma m_-},$$

$$a_1 = b_1 + \frac{\left(\frac{m_-\gamma}{1-\gamma} + 1\right)\frac{1}{K} + \frac{m_- - 1}{r} - \frac{m_-}{\rho(1-\gamma)}}{m_+ - m_-} \alpha_2^{1-m_+} L^{1-\gamma+\gamma m_+},$$

$$c_2 = b_2 + \frac{\frac{2-m_+}{4(\rho-2r+\theta^2)} + \frac{1-m_+}{2r} + \frac{m_+}{4\rho}}{(m_+ - m_-)R} \alpha_1^{1-m_-}.$$

Value Function $U(x)$

Legendre Transform Inverse Formula

$$U(x) = \min_{y>0} \left(\phi(y) + yx\right), \tag{5}$$

for any $x \in (0, \infty)$.

Optimization Problem

The agent's optimization problem is to maximize her/his expected utility

$$J(x; c, g, \pi, \tau) := \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left\{\alpha_1\left(c_t - Rc_t^2\right) + \alpha_2\frac{(g_t + L)^{1-\gamma}}{1-\gamma} - l\mathbf{1}_{\{0\le t<\tau\}}\right\}dt\right]$$

subject to the budget constraint

$$\mathbb{E}\left[\int_0^\tau H_t(c_t + g_t)dt + H_\tau\left(X_\tau + \frac{\epsilon}{r}\right)\right] \le x + \frac{\epsilon}{r}.$$

Here $\rho > 0$ is a subjective discount factor, $\alpha_i \geq 0$ $(i = 1, 2)$ with $\alpha_1 + \alpha_2 = 1$, $R > 0$, $L \geq 0$, $\gamma > 0$ $(\gamma \neq 1)$ is an agent's coefficient of relative risk aversion, and $l > 0$ is disutility from labor.

$$\begin{aligned}
V(x) &= \max_{(c,g,\pi,\tau)\in\mathcal{A}(x)} J(x; c, g, \pi, \tau)\\
&= \max_{(c,g,\pi,\tau)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\tau e^{-\rho t}\left\{\alpha_1\left(c_t - Rc_t^2\right) + \alpha_2\frac{(g_t + L)^{1-\gamma}}{1-\gamma} - l\right\}dt\right.\\
&\quad \left. + \int_\tau^\infty e^{-\rho t}\left\{\alpha_1\left(c_t - Rc_t^2\right) + \alpha_2\frac{(g_t + L)^{1-\gamma}}{1-\gamma}\right\}dt\right]\\
&= \max_{(c,g,\pi,\tau)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\tau e^{-\rho t}\left\{\alpha_1\left(c_t - Rc_t^2\right) + \alpha_2\frac{(g_t + L)^{1-\gamma}}{1-\gamma} - l\right\}dt + e^{-\rho\tau}U(X_\tau)\right].
\end{aligned}$$

For a Lagrange multiplier $\lambda > 0$, we obtain

$$\begin{aligned}\tilde{J}(\lambda;\tau) &= \max_{(c,g,\pi)\in\Pi_\tau(x)} \left[\mathbb{E}\left[\int_0^\tau e^{-\rho t}\left\{\alpha_1\left(c_t - Rc_t^2\right) + \alpha_2\frac{(g_t+L)^{1-\gamma}}{1-\gamma} - l\right\}dt + e^{-\rho\tau}U(X_\tau)\right]\right.\\
&\quad \left. -\lambda\mathbb{E}\left[\int_0^\tau H_t(c_t+g_t)dt + H_\tau\left(X_\tau + \frac{\epsilon}{r}\right)\right]\right]\\
&= \max_{(c,g,\pi)\in\Pi_\tau(x)} \mathbb{E}\left[\int_0^\tau e^{-\rho t}\left\{\alpha_1\left(c_t - Rc_t^2 - \lambda e^{\rho t}H_t\frac{c_t}{\alpha_1}\right)\right.\right.\\
&\quad \left. +\alpha_2\left(\frac{(g_t+L)^{1-\gamma}}{1-\gamma} - \lambda e^{\rho t}H_t\frac{g_t}{\alpha_2}\right) - l\right\}\\
&\quad \left. +e^{-\rho\tau}\left\{U(X_\tau) - \lambda e^{\rho\tau}H_\tau\left(X_\tau + \frac{\epsilon}{r}\right)\right\}\right]\\
&= \mathbb{E}\left[\int_0^\tau e^{-\rho t}\left\{\alpha_1\tilde{u}_1(y_t) + \alpha_2\tilde{u}_2(y_t) - l\right\} + e^{-\rho\tau}\tilde{U}(y_\tau)\right], \text{ where } y_t = \lambda e^{\rho t}H_t,\end{aligned}$$

$$\widetilde{u}_1(y) = \max_{c\ge 0}\left\{c - Rc^2 - \frac{1}{\alpha_1}yc\right\} = \frac{\left(1 - \frac{1}{\alpha_1}y\right)^2}{4R}\mathbf{1}_{\{0<y\le\alpha_1\}} + 0\cdot\mathbf{1}_{\{y>\alpha_1\}},$$

$$\begin{aligned}\widetilde{u}_2(y) &= \max_{g\ge 0}\left\{\frac{(g+L)^{1-\gamma}}{1-\gamma} - \frac{1}{\alpha_2}yg\right\} = \left\{\frac{\gamma}{1-\gamma}\left(\frac{1}{\alpha_2}y\right)^{-\frac{1-\gamma}{\gamma}} + L\frac{1}{\alpha_2}y\right\}\mathbf{1}_{\{0<y\le\alpha_2L^{-\gamma}\}}\\
&\quad + \frac{L^{1-\gamma}}{1-\gamma}\mathbf{1}_{\{y>\alpha_2L^{-\gamma}\}},\end{aligned}$$

$$\tilde{U}(y) = \max_{x>-\epsilon/r}\left[U(x) - y\left(x + \frac{\epsilon}{r}\right)\right] = U(I(y)) - yI(y) - \frac{\epsilon}{r}y \text{ with } I(\cdot) = (U')^{-1}(\cdot).$$

$$\tilde{V}(\lambda) = \sup_{\tau \in \mathcal{S}} \tilde{J}(\lambda; \tau)$$

Let

$$\phi(t, y) = \sup_{\tau} \mathbb{E}^{y_t = y} \left[\int_t^{\tau} e^{-\rho s} \{\alpha_1 \tilde{u}_1(y_s) + \alpha_2 \tilde{u}_2(y_s) - I\} + e^{-\rho\tau} \tilde{U}(y_\tau) \right].$$

Variational Inequality

Find a free boundary $\bar{y} > 0$ and a function $\phi(\cdot, \cdot) \in C^1((0, \infty) \times \mathbb{R}^+) \cap C^2((0, \infty) \times (\mathbb{R}^+ \setminus \{\bar{y}\}))$ satisfying

(1) $\mathcal{L}\phi(t, y) + e^{-\rho t} \{\alpha_1 \tilde{u}_1(y) + \alpha_2 \tilde{u}_2(y) - I\} = 0, \quad \bar{y} < y,$

(2) $\mathcal{L}\phi(t, y) + e^{-\rho t} \{\alpha_1 \tilde{u}_1(y) + \alpha_2 \tilde{u}_2(y) - I\} \leq 0, \quad 0 < y \leq \bar{y},$

(3) $\phi(t, y) \geq e^{-\rho t} \tilde{U}(y), \quad y \geq \bar{y},$

(4) $\phi(t, y) = e^{-\rho t} \tilde{U}(y), \quad 0 < y \leq \bar{y},$

for all $t > 0$, where

$$\mathcal{L} := \frac{\partial}{\partial t} + (\rho - r) y \frac{\partial}{\partial y} + \frac{1}{2} \theta^2 y^2 \frac{\partial^2}{\partial y^2}.$$

Proposition

We derive

$$\phi(y)=\begin{cases} a_1 y^{m_+}+\frac{\gamma}{1-\gamma}\left(\frac{1}{\alpha_2}\right)^{-\frac{1}{\gamma}}\frac{1}{K}y^{-\frac{1-\gamma}{\gamma}} \\ \quad -\frac{1}{4\alpha_1(\rho-2r+\theta^2)R}y^2+\frac{2LR-1}{2rR}y+\frac{\alpha_1}{4\rho R}, & \text{if } 0<y\leq\bar{y} \\ A_1 y^{m_+}+A_2 y^{m_-}+\frac{\gamma}{1-\gamma}\left(\frac{1}{\alpha_2}\right)^{-\frac{1}{\gamma}}\frac{1}{K}y^{-\frac{1-\gamma}{\gamma}} \\ \quad -\frac{1}{4\alpha_1(\rho-2r+\theta^2)R}y^2+\frac{2LR-1}{2rR}y+\frac{\alpha_1}{4\rho R}-\frac{l}{\rho}, & \text{if } \bar{y}<y\leq\alpha_2 L^{-\gamma} \\ B_1 y^{m_+}+B_2 y^{m_-}-\frac{1}{4\alpha_1(\rho-2r+\theta^2)R}y^2 \\ \quad -\frac{1}{2rR}y+\frac{\alpha_1}{4\rho R}+\frac{\alpha_2 L^{1-\gamma}}{\rho(1-\gamma)}-\frac{l}{\rho}, & \text{if } \alpha_2 L^{-\gamma}<y\leq\alpha_1 \\ C_2 y^{m_-}+\frac{\alpha_2 L^{1-\gamma}}{\rho(1-\gamma)}-\frac{l}{\rho}, & \text{if } y>\alpha_1, \end{cases}$$

$$\bar{y}=\frac{rl}{\rho\epsilon}\frac{m_-}{m_--1},\quad A_2=\left(\frac{l}{\rho}-\frac{\epsilon}{r}\bar{y}\right)\bar{y}^{-m_-}$$

$$B_2=A_2+\frac{\left(\frac{m_+\gamma}{1-\gamma}+1\right)\frac{1}{K}+\frac{m_+-1}{r}-\frac{m_+}{\rho(1-\gamma)}}{m_+-m_-}\alpha_2^{1-m_-}L^{1-\gamma+\gamma m_-},$$

$$B_1=\frac{\frac{2-m_-}{4(\rho-2r+\theta^2)}+\frac{1-m_-}{2r}+\frac{m_-}{4\rho}}{(m_+-m_-)R}\alpha_1^{1-m_+},\quad C_2=B_2+\frac{\frac{2-m_+}{4(\rho-2r+\theta^2)}+\frac{1-m_+}{2r}+\frac{m_+}{4\rho}}{(m_+-m_-)R}\alpha_1^{1-m_-},$$

$$A_1=B_1+\frac{\left(\frac{m_-\gamma}{1-\gamma}+1\right)\frac{1}{K}+\frac{m_--1}{r}-\frac{m_-}{\rho(1-\gamma)}}{m_+-m_-}\alpha_2^{1-m_+}L^{1-\gamma+\gamma m_+}.$$

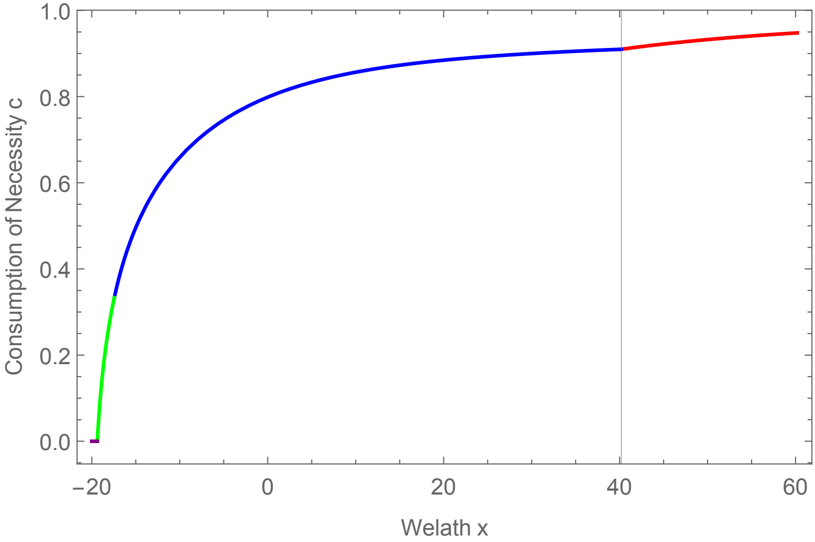
Value Function $V(x)$

Legendre Transform Inverse Formula

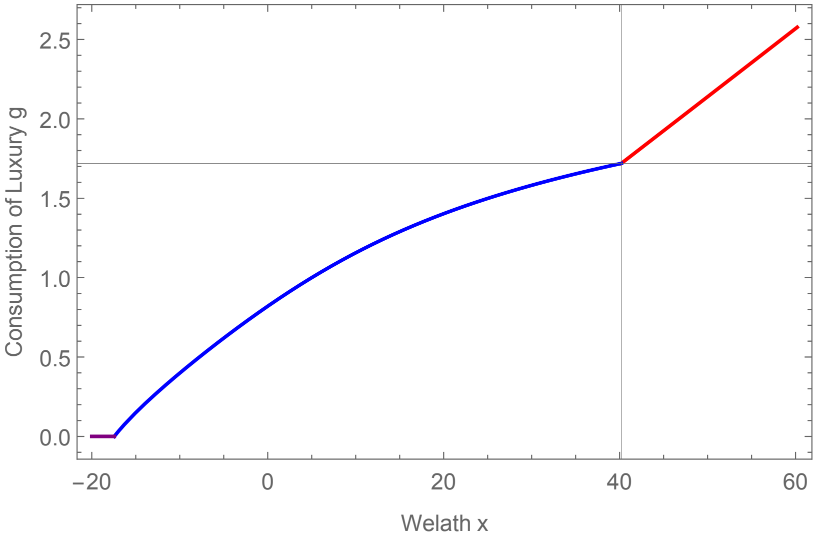
$$V(x) = \min_{y>0} \left[\phi(y) + y \left(x + \frac{\epsilon}{r} \right) \right], \tag{6}$$

for any $x \in (-\epsilon/r, \infty)$.

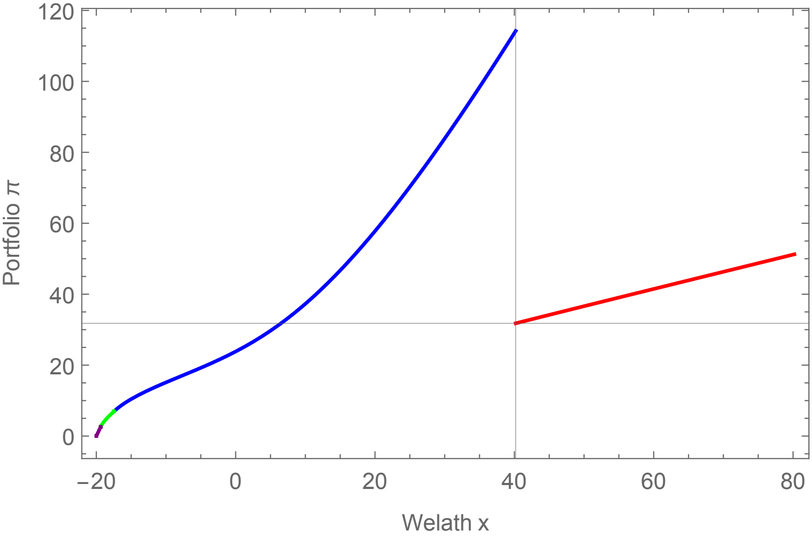
Optimal Consumption of Necessity



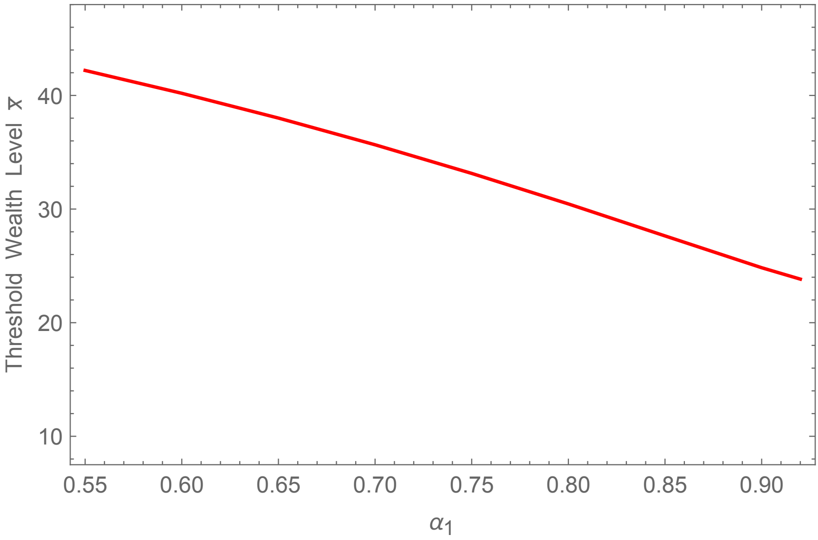
Optimal Consumption of Luxury



Optimal Portfolio



Threshold Wealth Level



Concluding Remarks

- We investigate an optimal consumption, gift, investment, and voluntary retirement choice model of an agent who has a motive for giving by using the weighted sum of a quadratic utility function and a HARA utility function.
- We use the martingale method to derive closed form solutions for optimal consumption, gift, and investment.
- We also give some numerical implications.

Thank you for your attention!!