Optimal Order Exposure in a Limit Order Market

Xuefeng Gao

Systems Engineering & Engineering Management
The Chinese University of Hong Kong

Joint work with Yuanyuan Chen (NUS) and Duan Li (CUHK)

5th Asian Quantitative Fianance Conference, Seoul

April 25, 2017

Outline

Introduction

Problem Description

Theoretical Results

Empirical Study

Summary

Outline

Introduction

Problem Description

Theoretical Results

Empirical Study

Summary

Limit Order Books

In limit order markets, investors can submit two basic order types:

- Market order: filled immediately at the best available price.
- Limit order: an order to trade at a specified price or better;
  - Limit orders are displayed to other participants and wait for execution in the limit order book.

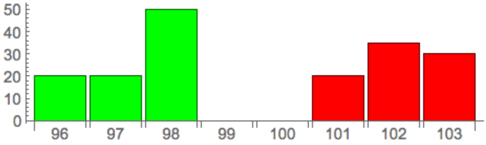


Figure: A Limit Order Book

Best bid price: 98. Best ask price: 101. Bid-ask Spread: 3.

Exposure Risk of Displayed Limit Orders

- Displayed limit orders, especially large ones, could reveal trading intentions to other participants in the market, and adversely impact the price.
  - Limit orders may contain fundamental information of the traded asset (Kaniel and Liu (2006); Cao et al. (2009)).
  - Visible limit orders can induce front running and liquidity competition (Harris (1997); Buti and Rindi (2013)).
  - Empirical studies on price impact of limit orders: Hautsch and Huang (2012a); Eisler et al. (2012); Cont et al. (2014).

Exposure Risk of Displayed Limit Orders

- Displayed limit orders, especially large ones, could reveal trading intentions to other participants in the market, and adversely impact the price.
  - Limit orders may contain fundamental information of the traded asset (Kaniel and Liu (2006); Cao et al. (2009)).
  - Visible limit orders can induce front running and liquidity competition (Harris (1997); Buti and Rindi (2013)).
  - Empirical studies on price impact of limit orders: Hautsch and Huang (2012a); Eisler et al. (2012); Cont et al. (2014).
- Such signalling or exposure risk of displayed limit orders is a significant concern for market participants as the average order size in equity markets is small.
  - (Institutional) Investors wish to hide large-size orders to avoid being seen and exploited by other traders.

Hidden Orders

- Many exchanges (NYSE, NASDAQ, LSE, ASX etc) offer hidden order types to allow traders to "hide" all or a portion of the size of their orders on the book.
  - Iceberg/Reserve order: a limit order with a peak portion displayed, while the remaining quantity is hidden.
  - (Completely) hidden order: a limit order which does not display any order quantity.

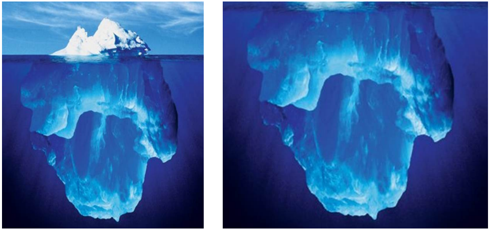


Figure: Source: spindrift-racing.com

Popularity of Hidden Orders

- Hidden orders are popular in trading in equity markets.
  - Bloomfield et al. (2015): approximately 20% of marketable orders are executed against hidden orders in U.S. markets.
  - Boulatov and George (2013): more than 15% of order flows in NASDAQ are non-displayed.
  - Bessembinder et al. (2009): 44% of the order volume are hidden in a sample of 100 stocks traded on Euronext-Paris.
- Driven in part by the rise of crossing networks/dark pools and in part by competitive pressures from new exchanges and trading platforms.
- This popularity also indicates that investors have a strong need to hide their orders when they are trading on lit venues.

Research Question

- Optimal order submission strategies in limit order markets where traders have an option to use hidden orders.
- Specifically, we focus on an optimal order placement problem among limit and hidden orders, and we call it "optimal order exposure problem"
  - Hide or not? sizes to hide?
- Relevant for trading desks responsible for executing large block orders received from portfolio managers.

Hidden Orders vs (Displayed) Limit Orders

- Benefit: hidden orders reduce the exposure/signalling risk, and incur little price impact.
  - Empirical studies found that investors use hidden orders to reduce the exposure cost (Aitken et al. (2001), De Winne and D'hondt (2007) and Bessembinder et al. (2009)).
- Cost: hidden orders lose time priority with respect to incoming limit orders at the same limit price.
  - Execution priority: price – visibility – time
  - Hiding an order lowers the execution probability.

Related Literature on hidden orders

- A few empirical works have been done to analyze order exposure strategies, including why, when, and where traders place hidden orders
  - Hautsch and Huang (2012b), Bessembinder et al. (2009) and D'Hondt et al. (2004))
- Few theoretical models on order exposure strategies.
  - Market equilibrium: Moinas (2010), Buti and Rindi (2013).
  - Static/singe-stage setting: Esser and Mönch (2007), Cebiroglu and Horst (2013).
- Our paper: Study a multi-stage optimal order exposure problem using dynamic programming.

Outline

Introduction

Problem Description

Theoretical Results

Empirical Study

Summary

Problem Description

- An investor aims to buy n_1 shares of an asset over a fixed time horizon which is split into $T-1$ uniform time periods.
- At each stage $t = 1, ..., T-1$, she decides to submit $l_t \geq 0$ shares of limit orders and $h_t \geq 0$ shares of hidden orders, both pegging at best bid price.
- If the target n_1 is not reached by time T, the trader buys the remaining shares using a market order at time T with a fee f per share.
- Optimization problem: find the optimal trading sequence $(l_t^*, h_t^*)_{t=1,...,T-1}$ to minimize the total expected trading cost.

Model setup

1. In the absence of this trader's orders, the best ask price follows $A_{t+1} = A_t + \epsilon_t$, where $\mathbb{E}[\epsilon_t] = 0$ and $Cov(\epsilon_t, \epsilon_k) = 0$ for $k \neq t = 1, \ldots, T-1$.
2. The bid-ask spread is a constant (one tick).
3. During each time period [t,t+1),
   - p_l: fill probability of the limit order;
   - p_h: fill probability of hidden orders given that the limit order is executed;
   - p_d: the probability of successful detection of hidden orders by other traders after hidden orders' execution.
   - $0 < p_l, p_h, p_d < 1$.
   - Partial fills of orders are not considered.
4. At the end of each period, the trader cancels non-executed orders and submit orders with sizes (l_{t+1}, h_{t+1}) for the next period.

Assumptions: linear price impact (Almgren-Chriss)

To capture the exposure/signalling risk of limit orders, we assume

- Limit orders l_t incur temporary price impact cl_t: the execution price of this order is $A_t - 1 + cl_t$.
- Executed limit orders generate permanent price impact αl_t, which affects the ask price at next stage.
- Hidden orders have no price impact unless they are executed and detected, which incur linear permanent impact βh_t.

We also assume

- A buy market order of size x will push up the market price by γx.
- $\gamma > \alpha \geq c \geq 0$.

Dynamics of state variables (n_t, A_t)

The dynamics of state variables, the remaining shares to buy $n_t (\geq 0)$ and the best ask price A_t satisfies:

$$(n_{t+1}, A_{t+1}) = \begin{cases} (n_t, A_t + \epsilon_t) & \text{with prob. } 1 - p_l, \\ (n_t - l_t, A_t + \alpha l_t + \epsilon_t) & \text{with prob. } p_l(1 - p_h), \\ (n_t - l_t - h_t, A_t + \alpha l_t + \epsilon_t) & \text{with prob. } p_l p_h (1 - p_d), \\ (n_t - l_t - h_t, A_t + \alpha l_t + \beta h_t + \epsilon_t) & \text{with prob. } p_l p_h p_d, \end{cases}$$

where $(l_t, h_t) \in \{l_t \geq 0, h_t \geq 0, l_t + h_t \leq n_t\}$ is the number of limit and hidden orders to submit at stage $t = 1, \cdots, T-1$.

Modelling and Optimization

- Denote $\tilde{C}_t(n_t, A_t, l_t, h_t)$ for the expected execution cost of placing l_t shares of limit orders and h_t shares of hidden orders during $[t, t+1)$ with ask price A_t and remaining share to buy n_t.
- The investor solves the dynamic optimization problem:

$$\begin{aligned} \min_{(l_t,h_t)_{t=1}^{T-1}} \quad & \mathbb{E}[\sum_{t=1}^{T-1} \tilde{C}_t(n_t, A_t, l_t, h_t) + V_T(n_T, A_T)] \\ s.t. \quad & h_t + l_t \leq n_t, \;\; h_t, l_t \geq 0, \end{aligned}$$

where

$$V_T(n_T, A_T) = n_T(A_T + \gamma n_T + f).$$

Properties

- We can show that the optimal strategy satisfies: $l_t^* + h_t^* = n_t$ for $1 \leq t \leq T-1$.
- Then we can derive the dynamic programming recursion for the value function $V_t(n_t, A_t)$:

$$\begin{aligned} V_t(n_t, A_t) &= n_t A_t + \min_{0 \leq l_t \leq n_t} C_t(n_t, l_t), \quad \text{for } 1 \leq t \leq T-1, \\ C_t(n_t, l_t) &= p_l l_t(c l_t - 1) + p_l p_h (n_t - l_t)(c l_t - 1) \\ &+ p_l(1 - p_h) \cdot V_{t+1}(n_t - l_t, \alpha l_t) \\ &+ (1 - p_l) \cdot V_{t+1}(n_t, 0), \end{aligned} \tag{1}$$

with the terminal cost

$$V_T(n_T, A_T) = n_T A_T + \gamma n_T^2 + f n_T.$$

Outline

Introduction

Problem Description

Theoretical Results

Empirical Study

Summary

Multi-stage Dynamic Programming

- Situations where the optimal strategy is of a bang-bang type

Theorem

Under the following market condition (C1)

$$c \leq (1 - p_h)(\alpha - c),$$

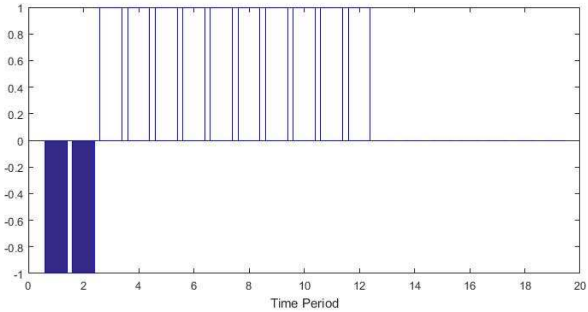
the optimal strategy is of a bang-bang type,

$$I_t^*(n_t) = \begin{cases} 0 & \text{for } t = 1, \cdots, t_h, \\ n_t & \text{for } t = t_h + 1, \cdots, T - 1, \end{cases}$$

where t_h ('length-of-hiding') can be found explicitly. In addition, the value function $V_t(n_t, A_t)$ can be expressed in closed form and it is quadratic in n_t.

Structure of Optimal Strategy under (C1)

- Under market condition (C1), we can infer from the obtained analytical solution that the optimal strategy is to submit pure hidden order in the early time periods and submit pure limit order later.



Multi-stage Model

- Market condition (C1) is not satisfied

When (C1) is not satisfied, we need an assumption to derive a closed form solution to the constrained optimal control problem.

Assumption

$$g^{(T-2)}(\gamma) > \alpha - c,$$

where $g^{(T-2)}(\cdot)$ is $(T-2)-$th iterate of $g(\cdot)$ defined by

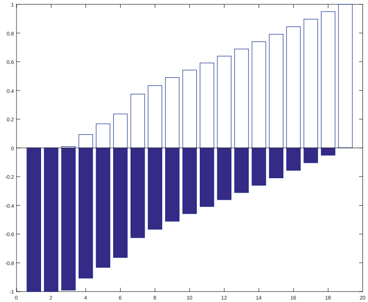
$$g(x) = p_l c + (1-p_l)x - p_l \frac{[c-(1-p_h)(\alpha - c)]^2}{4(1-p_h)(x+c-\alpha)}.$$

- A sufficient condition for the (strictly) convexity the cost-to-go function $C_t(n_t, l_t)$ in terms of the decision variable l_t, for each $t = 1, \ldots, T-1$.

Structure of the Optimal Trading Strategy

- Market condition (C1) is not satisfied

The optimal trading sequence has the following structure.



- Submit pure hidden orders in the beginning, turn to a mixture of limit and hidden orders later, and finally pure limit orders when the deadline approaches.

Optimal Trading Strategy

- Market condition (C1) is not satisfied

Theorem

Suppose $c > (1 - p_h)(\alpha - c)$. Under the previous assumption, the optimal control $l_t^*(n_t)$ at stage t is a continuous and piecewise linear function of n_t:

$$l_t^*(n_t) = \begin{cases} n_t & \text{for } n_t < N_{t,0}^c, \\ a_0 n_t + b_0 & \text{for } n_t \in [N_{t,0}^c, N_{t,1}^c), \\ \quad \vdots & \quad \vdots \\ a_{j_t} n_t + b_{j_t} & \text{for } n_t \in [N_{t,j_t}^c, N_{t,j_t+1}^c), \\ 0 & \text{for } n_t \in [N_{t,j_t+1}^c, \infty), \end{cases}$$

In addition, the optimal value function $V_t(n_t, A_t)$ is piecewise quadratic and continuously differentiable in n_t.

Outline

Introduction

Problem Description

Theoretical Results

Empirical Study

Summary

Main Task and Model Parameters

- We want to test the convexity assumption:

$$g^{(T-2)}(\gamma) > \alpha - c,$$

where

$$g(x) = p_l c + (1 - p_l)x - p_l \frac{[c - (1 - p_h)(\alpha - c)]^2}{4(1 - p_h)(x + c - \alpha)}.$$

- Estimating model parameters:
  - p_l: fill probability of limit orders pegged at the best price
  - p_h: fill probability of hidden orders pegged at the best price conditional on the execution of limit orders at the same price level
  - γ: price impact of market orders
  - α: permanent price impact of limit orders
  - c: temporary price impact of limit orders

Data

We use the NASDAQ's TotalView-ITCH data containing message data of order events from LOBSTER.

- Tick data of three stocks, Intel (INTC), Microsoft (MSFT) and General Electric Company (GE), from 10:00 a.m. to 4:00 p.m. for the whole June in the year of 2012.

Stock Symbol	INTC	MSFT	GE
Aver. num. of events per sec.	19.2655	22.4786	14.4251
Aver. bid-ask spread	$ 0.0122	$ 0.0127	$ 0.0130
Aver. mid-price	$ 26.356	$ 29.640	$ 19.429
Aver. num. of mid-price change per min.	8.320	9.359	4.240
Aver. trade size (in shares)	309.226	350.789	430.519

Table: Descriptive Statistics of Three Stocks

Methodologies to Estimate the Parameters

- Fill probability of limit orders p_l: we apply survival analysis to estimate p_l (see Lo et al. (2002)).
- Due to limitation of data, we set $p_h = p_l$ in the empirical study and vary p_h to test the generality.
- Price impact of market orders γ: we calculate the average best ask price change induced by a buy market order through paper trades.
- Permanent price impact of limit orders: we estimate α with a vector autoregressive model (see Hautsch and Huang (2012a)).
- Temporary price impact of limit orders: we estimate c by the average price change before the order is fully executed.

Estimation results

Time Period	$\bar{p}_l$	$99\% CI$	γ	α	c	Number
10:00-10:30	0.9035	[0.8598,0.9381]	0.46	0.31	0.24	25211
10:30-11:00	0.8100	[0.7593,0.8558]	0.38	0.25	0.16	15354
11:00-11:30	0.6514	[0.6082,0.6944]	0.36	0.22	0.14	12523
11:30-12:00	0.6606	[0.6031,0.7174]	0.36	0.28	0.17	11188
12:00-12:30	0.5637	[0.5052,0.6238]	0.35	0.21	0.14	8252
12:30-13:00	0.2966	[0.2585,0.3388]	0.26	0.09	0.07	7206
13:00-13:30	0.4971	[0.4512,0.5450]	0.27	0.16	0.10	10418
13:30-14:00	0.6725	[0.6107,0.7331]	0.30	0.15	0.09	12716
14:00-14:30	0.6430	[0.5917,0.6942]	0.33	0.28	0.18	14457
14:30-15:00	0.8259	[0.7329,0.9012]	0.35	0.28	0.18	14015
15:00-15:30	0.8362	[0.7847,0.8813]	0.39	0.19	0.12	14538
15:30-16:00	0.8363	[0.8043,0.8658]	0.29	0.14	0.09	25713

Table: Parameters Estimation for MSFT on June 1st, 2012.

Test the Convexity Assumption and results

We estimate the parameters with separate half-hour subsamples (Cont et al. (2014)).

- Applying the estimated parameters to test the convexity assumption for T varying from three to twenty, our convexity assumption holds in more than 97% of the subsamples.
- Varying the parameters within their confidence intervals or possible ranges for each subsample, the assumption still holds in more than 90% of the subsamples.

The convexity assumption is not restrictive in general. So our closed-form solution scheme can be applied to these stocks studied.

Outline

Introduction

Problem Description

Theoretical Results

Empirical Study

Summary

Summary

- Hidden orders are among the most popular order types in trading across many stock exchanges.
- We derive closed form solutions and structural results for a multi-stage optimal order exposure problem under a certain assumption.
- We calibrate the model using NASDAQ historical data and demonstrate the generality of the assumption.
- Our results suggest that patient traders executing large-size orders tend to use more hidden orders.
- Our results could be useful for investors executing large orders when they have an option of hiding their orders.

Thank You! Questions?

Single-stage Model: $T = 2$

- For a single-stage model, the cost-to-go function in (1) becomes

$$\begin{aligned} & C_1(n_1, l) \\ = \quad & p_l(1-p_h)(\gamma-\alpha+c)\cdot l^2 \\ + \quad & [(p_l p_h c + p_l(1-p_h)(\alpha-2\gamma))n_1 - p_l(1-p_h)(f+1)]\cdot l \\ + \quad & (1-p_l p_h)(\gamma n_1^2 + f n_1) - p_l p_h n_1. \end{aligned}$$

- As $\gamma > \alpha - c$, the optimization problem $\min_{0 \leq l \leq n_1} C_1(n_1, l)$ becomes a strictly convex quadratic programming problem with one decision variable $l \in [0, n_1]$.

Single-stage Model: analytical solution

For the single-stage model, the optimal strategy is:

$$I^*(n_1) = \begin{cases} n_1 & \text{if} \quad n_1 < N_1, \\ I^G(n_1, \gamma, f) & \text{if} \quad n_1 \in [N_1, N_2], \\ 0 & \text{if} \quad n_1 > N_2, \end{cases}$$

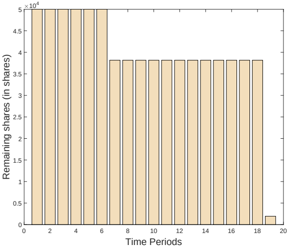
where the pair of cutting points (N_1, N_2) is given by

$$(N_1, N_2) = \begin{cases} (\infty, \infty), & \text{if } c \le (1-p_h)(\alpha - c), \\ \left(\frac{(1-p_h)(f+1)}{c-(1-p_h)(\alpha-c)}, \frac{(1-p_h)(f+1)}{c-(1-p_h)(2\gamma+c-\alpha)} \right), & \text{if } c > (1-p_h)(2\gamma + c - \alpha), \\ \left(\frac{(1-p_h)(f+1)}{c-(1-p_h)(\alpha-c)}, \infty \right), & \text{otherwise}, \end{cases}$$

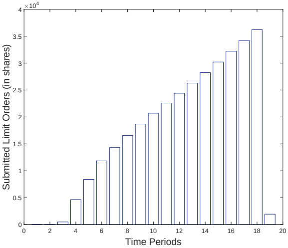
and

$$I^G(n_1, \gamma, f) = \left(1 - \frac{c-(1-p_h)(\alpha-c)}{2(1-p_h)(c+\gamma-\alpha)}\right) \cdot n_1 + \frac{f+1}{2(c+\gamma-\alpha)}.$$

Optimal Trading Sequence



(a) Remaining shares n_t



(b) Optimal limit order size l_t^*

Figure: Optimal Strategy Sequence. For each stage t, the optimal strategy suggests the sizes of limit and hidden orders to be submitted. The remaining shares n_t and the sizes of submitted limit orders, l_t^*, are illustrated in the left and right figures, respectively.

Bibliography I

Aitken, M. J., H. Berkman, and D. Mak (2001). The use of undisclosed limit orders on the australian stock exchange. *Journal of Banking & Finance 25*(8), 1589–1603.

Bessembinder, H., M. Panayides, and K. Venkataraman (2009). Hidden liquidity: an analysis of order exposure strategies in electronic stock markets. *Journal of Financial Economics 94*(3), 361–383.

Boulatov, A. and T. J. George (2013). Hidden and displayed liquidity in securities markets with informed liquidity providers. *Review of Financial Studies 26*(8), 2096–2137.

Buti, S. and B. Rindi (2013). Undisclosed orders and optimal submission strategies in a limit order market. *Journal of Financial Economics 109*(3), 797–812.

Cebiroglu, G. and U. Horst (2013). Optimal order exposure and the market impact of limit orders. *Available at SSRN 1997092*.

Cont, R., A. Kukanov, and S. Stoikov (2014). The price impact of order book events. *Journal of Financial Econometrics 12*(1), 47–88.

Bibliography II

De Winne, R. and C. D'hondt (2007). Hide-and-seek in the market: placing and detecting hidden orders. *Review of Finance 11*(4), 663–692.

D'Hondt, C., R. De Winne, and A. Francois-Heude (2004). Hidden orders on euronext: Nothing is quite as it seems. *Available at SSRN 379362*.

Eisler, Z., J.-P. Bouchaud, and J. Kockelkoren (2012). The price impact of order book events: market orders, limit orders and cancellations. *Quantitative Finance 12*(9), 1395–1419.

Esser, A. and B. Mönch (2007). The navigation of an iceberg: The optimal use of hidden orders. *Finance Research Letters 4*(2), 68–81.

Harris, L. (1997). Order exposure and parasitic traders. *University of Southern California working paper*.

Hautsch, N. and R. Huang (2012a). The market impact of a limit order. *Journal of Economic Dynamics and Control 36*(4), 501–522.

Hautsch, N. and R. Huang (2012b). On the dark side of the market: identifying and analyzing hidden order placements. *Available at SSRN 2004231*.

Bibliography III

Lo, A. W., A. C. MacKinlay, and J. Zhang (2002). Econometric models of limit-order executions. *Journal of Financial Economics 65*(1), 31–71.

Moinas, S. (2010). Hidden limit orders and liquidity in order driven markets. *TSE Working Paper 10*.