Recursive Utility with Investment Gains and Losses: Existence, Uniqueness, and Convergence

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The Fifth Asian Quantitative Finance Conference
Seoul

April 24, 2017

Based on a joint work with Jing Guo (Columbia)

Recursive Utility

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- The recursive utility process $\{V_t\}$ is defined recursively:

$$V_t = H(C_t, \mathcal{M}_t(V_{t+1})), \quad t = 0, 1, \dots,$$

where $\mathcal{M}_t(X)$ stands for the *certainty equivalent* of random quantity X conditioning on the information at time t and $H(c, z)$ is an *aggregator function*.

Constant EIS and RRAD

- The most commonly used aggregator function and certainty equivalent in the literature are

$$H(c,z):=\begin{cases}[(1-\beta)c^{1-\rho}+\beta z^{1-\rho}]^{\frac{1}{1-\rho}}, & 0<\rho\neq 1,\\ e^{(1-\beta)\ln c+\beta\ln z}, & \rho=1,\end{cases}$$

$$\mathcal{M}_t(X):=u^{-1}\big(\mathbb{E}_t[u(X)]\big),\quad u(x):=\begin{cases}x^{1-\gamma}/(1-\gamma), & 0<\gamma\neq 1,\\ \ln(x), & \gamma=1,\end{cases},$$

where $\beta\in(0,1)$ is a *discount rate*, $1/\rho$ is the elasticity of intertemporal substitution (EIS), and γ is the relative risk aversion degree (RRAD).

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- The existence and uniqueness of the recursive utility process, however, have not been fully established.

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    - At time t, an agent consumes C_t and invests dollar amount $\Theta_{i,t}$ in asset i, $i = 0, 1, \ldots, n$, where asset 0 is a risk-free asset and the other assets are risky. The agent's utility process $\{U_t\}$ is defined recursively as follows:

$$U_t = H\left(C_t, \mathcal{M}_t(U_{t+1}) + \sum_{i=1}^{n} b_i G_{i,t}\right), \quad t = 0, 1, \ldots$$

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  - $b_i \geq 0$ is a constant, and $G_{i,t}$ stands for the utility of the *gain and loss* experienced by the agent for her investment in asset i. More precisely, in Barberis and Huang (2008, 2009) and Barberis et al. (2006),

$$G_{i,t} = \mathbb{E}_t\big[\Theta_{i,t}(R_{i,t+1} - R_{f,t+1})\mathbf{1}_{R_{i,t+1} > R_{f,t+1}} + k\Theta_{i,t}(R_{i,t+1} - R_{f,t+1})\mathbf{1}_{R_{i,t+1} < R_{f,t+1}}\big]$$

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- Several variants of the model of narrow framing have also been proposed in the literature; see De Giorgi and Legg (2012) and He and Zhou (2014)
- The existence and uniqueness of the utility process in the model of narrow framing and its variants, however, have never been studied.

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- When the utility of gains and losses is negative, however, the total utility can be non-existent or non-unique
- We prove that the total utility, when uniquely exists, can be computed by a recursive algorithm with any starting point.
- We consider a portfolio selection problem with narrow framing and solve it by proving that the corresponding dynamic programming equation has a unique solution

Literature on Recursive Utility with KP Aggregator and CE

	Setting of consumption process	Existence	Uniqueness	Convergence	Utility of Investment G&L	Dynamic Programming
Epstein and Zin (1989)	bounded growth rate	$\rho \neq 1$	N	specific starting point	N	N
Ozaki and Streufert (1996)	12 conditions	$\rho > 1$	$\rho > 1$	N	N	Y
Marinacci and Montrucchio (2010)	essentially constant growth rate	$\rho < 1$	$\rho < 1$	N	N	N
Hansen and Scheinkman (2012)	Markovian	$\rho \neq 1$ and $\gamma \neq 1$	$\frac{1-\gamma}{1-\rho} \geq 1$	Specific starting point	N	N
The present paper	Finite-state Markovian	any ρ, γ	any ρ, γ	any starting point	Y	Y

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- Using the homogeneity of H of degree one, we can write the recursive equation as

$$\frac{V_t}{c_t} = H\left(1, \mathcal{M}_t\left(A_{t+1}\frac{c_{t+1}}{c_t}\frac{V_{t+1}}{c_{t+1}}\right) + \frac{B_t}{c_t}\right), \quad t = 0, 1, \ldots,$$

so we study the existence and uniqueness of $\{V_t/c_t\}$ as a solution to this equation

Markovian Setting

Assumption

(i) $\{(X_t, Y_t)\}$ *is a Markov process and the joint distribution of* (X_{t+1}, Y_{t+1}) *conditioned on* (X_t, Y_t) *depends only on* X_t*.*

(ii) *Consumption dynamics evolve as*

$$\log(c_{t+1}) - \log(c_t) + \log A_{t+1} = \kappa(X_t, X_{t+1}, Y_{t+1}), \quad t = 0, 1, \ldots$$

for some real-valued measurable function κ*.*

(iii) $B_t/c_t = \varpi(X_t), t = 0, 1, \ldots$ *for some real-valued measurable function* ϖ*.*

(iv) *For any state* x, $\mathbb{E}_t\left[u\left(e^{\kappa(X_t, X_{t+1}, Y_{t+1})}\right)|X_t = x\right]$ *exists.*

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- Then, f becomes the fixed point of operator $\mathbb{T}$ defined as

$$\mathbb{T}f(x) := H\Bigg(1, u^{-1}\left(\mathbb{E}_t\left[u\left(e^{\kappa(X_t,X_{t+1},Y_{t+1})}f(X_{t+1})\right)|X_t = x\right]\right) + \varpi(x)\Bigg), \quad x \in \mathbb{X},$$

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- We do not impose any assumptions on $\{Y_t\}$.
- The classical Brouwer fixed point theorem does not apply.

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- To this end, consider the following operator

$$\mathbb{U}h(x) := \mathbb{E}_t \left[u\left(e^{\kappa(X_t, X_{t+1}, Y_{t+1})} \right) h(X_{t+1}) | X_t = x \right], \quad x \in \mathbb{X}.$$

Perron-Frobenius

Proposition

(i) *Suppose* $\gamma \neq 1$. *Then, there exist* $\eta > 0$ *and* $v \in \mathcal{X}_{++}$ *such that*

$$\mathbb{U}v(x) = \eta v(x), \quad x \in \mathbb{X}.$$

Moreover, η *and* v *are the Perron-Frobenius eigenvalue and eigenvector of* $\tilde{\mathbf{P}}$*, respectively.*

(ii) *Suppose* $\gamma = 1$. *Then, there exist* $\eta \in \mathbb{R}$ *and* $v \in \mathcal{X}$ *such that*

$$\mathbb{E}_t[\kappa(X_t, X_{t+1}, Y_{t+1}) | X_t = x] = -\mathbb{E}_t[v(X_{t+1}) | X_t = x] + v(x) + \eta.$$

In addition,

$$\eta = \sum_{x \in \mathbb{X}} \pi_x \mathbb{E}_t[\kappa(X_t, X_{t+1}, Y_{t+1}) | X_t = x],$$

where vector $(\pi_x)_{x \in \mathbb{X}}$ *is the stationary distribution of* $\{X_t\}$.

Perron-Frobenius (Cont'd)

Proposition

(iii) *Define* $\delta := u^{-1}(\eta)$. *Then,*

$$\delta = \max_{f \in \mathcal{X}_{++}} \min_{x \in \mathbb{X}} \frac{u^{-1}\left(\mathbb{E}_t\left[u\left(e^{\kappa(X_t, X_{t+1}, Y_{t+1})} f(X_{t+1})\right) | X_t = x\right]\right)}{f(x)}.$$

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δ is a variant of the certainty equivalent of $\exp[\kappa(X_t, X_{t+1}, Y_{t+1})]$.

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Part (i) of the Proposition is from Hansen and Scheinkman (2012) and parts (ii) and (iii) are new.

The Case in Which ϖ is Nonnegative

Theorem

Assume $\varpi(x) \geq 0$, $x \in \mathbb{X}$. *Recall* δ *as defined in the previous Proposition and assume* $\beta\delta^{1-\rho} < 1$. *Then, the fixed point of* $\mathbb{T}$ *in* $\mathcal{X}_{++}$ *uniquely exists. Moreover, for any* $f \in \mathcal{X}_{++}$, $\{\mathbb{T}^n f\}_{n\geq 0}$ *converges to the fixed point.*

The Case in Which ϖ is Nonnegative (Cont'd)

- It is necessary to restrict the domain of $\mathbb{T}$ to be $\mathcal{X}_{++}$; otherwise, uniqueness is not guaranteed.
- We proved the existence and uniqueness for any values of EIS ρ and RRAD γ.
- We also proved that the iterated sequence of applying the recursive equation repeatedly always converge to the total utility process for any starting point
- When $\varpi \equiv 0$, we recover the classical recursively utility, so our results generalize those of the existence and uniqueness of recursive utility in the literature.

Example: When ϖ is Negative

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- Assume $\beta\delta^{1-\rho} < 1$
- Consider $\varpi < 0$.
- The domain of T is then $[-\varpi/\delta, +\infty)$.

Example: When ϖ is Negative (Cont'd)

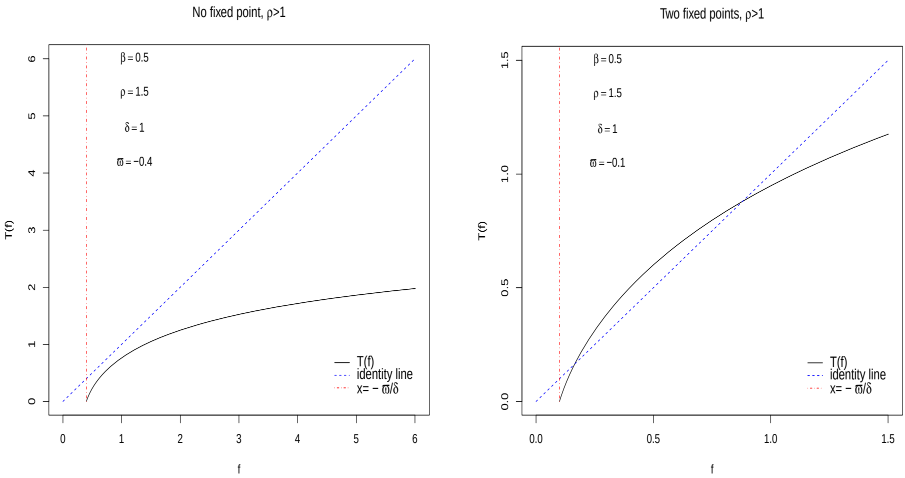

Figure: $T(f)$ when $\rho > 1$.

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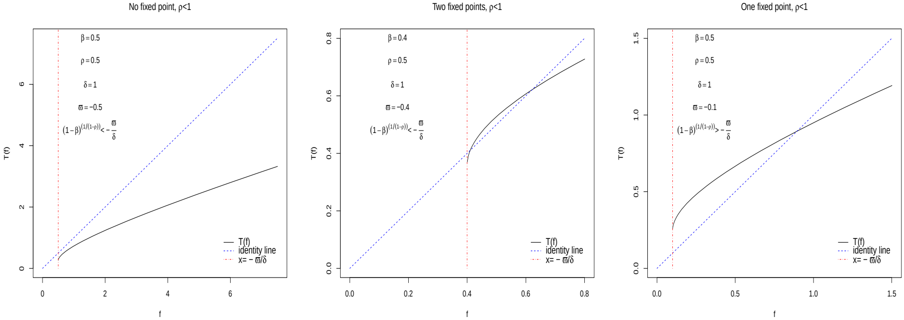

Figure: $T(f)$ when $\rho < 1$.

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Assumption

Denote

$$f_0(x) := H(1, \varpi^+(x)), \quad x \in \mathbb{X}.$$

Assume $\mathbb{T}f_0$ *is well defined, i.e.,*

$$u^{-1}\left[\mathbb{E}_t\left(u\left(e^{\kappa(X_t, X_{t+1}, Y_{t+1})} f_0(X_{t+1})\right) | X_t = x\right)\right] + \varpi(x) \geq 0, \quad x \in \mathbb{X},$$

and $\mathbb{T}^m f_0 > f_0$ *for some* $m \geq 1$.

Existence and Uniqueness When ϖ Is Negative (Cont'd)

Theorem

Suppose the previous assumption holds. Assume $\beta\delta^{1-\rho} < 1$. Then, the fixed point of $\mathbb{T}$ in its domain uniquely exists and is strictly larger than f_0 point-wisely. Moreover, for any f such that $\mathbb{T}f$ is well defined, sequence $\{\mathbb{T}^n f\}_{n\geq 0}$ converges to the fixed point of $\mathbb{T}$.

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- In Guo and He (2017), we propose a new model that accounts for narrow framing and, more generally, for utility of investment gains and losses such that
  - it leads to existence, uniqueness, and convergence of the total utility process; and
  - it is more tractable than the existing model in the literature.

Portfolio Selection with Narrow Framing

- We also consider portfolio selection with narrow framing.
- We prove that the value function of the portfolio selection satisfies a dynamic programing equation
- We show that the solution to the dynamic programming equation uniquely exists and any iterative sequence obtained by applying the equation repeatedly converges to the fixed point.

Portfolio Selection with Narrow Framing (Cont'd)

- More precisely, consider a market with n risky stocks and one risk-free asset.
- In the period t to $t+1$, stock i's gross return is $r_i(X_t, X_{t+1}, Y_{t+1})$ and the risk-free gross return is $r_0(X_t)$.
- At time t, an agent decides the optimal percentage of consumption $c(X_t)$ and allocation to stock $\boldsymbol{\theta}(X_t) := \big(\theta_1(X_t), \dots, \theta_n(X_t)\big)$.
- In consequence, his dollar consumption process is $C_t = c(X_t)W_t$, where W_t is the wealth process.
- The agent's preferences are modeled by the total utility process U_t, where

$$U_t = H\left(C_t, \mathcal{M}_t(U_{t+1}) + \sum_{i=1}^{n} b_i G_{i,t}\right), \quad t = 0, 1, \dots$$

- The objective is to maximize his total utility.

Portfolio Selection with Narrow Framing (Cont'd)

- Assume the feasible set to be

$$\mathcal{A} := \{(c, \boldsymbol{\theta}) | c(x) \in I_x, \boldsymbol{\theta}(x) \in J_x, x \in \mathbb{X}\},$$

  where for each x, I_x and J_x are compact.
- Denote $\Phi(X_t)$ as the optimal utility per unit wealth when the market state is at X_t.

Portfolio Selection with Narrow Framing (Cont'd)

- Dynamic programming equation:

$$\Phi(x) = \mathbb{W}\Phi(x), \quad x \in \mathbb{X},$$

where

$$\mathbb{W}\Phi(x) := \max_{\bar{c} \in I_x} H\left(\bar{c}, (1-\bar{c}) \max_{\bar{\boldsymbol{\theta}} \in J_x} D_\Phi(x, \bar{\boldsymbol{\theta}})\right), \quad x \in \mathbb{X},$$

$$D_\Phi(x, \bar{\boldsymbol{\theta}}) := \sum_{i=1}^{n} \bar{\theta}_i b_i g_i(x)$$

$$+ u^{-1}\left(\mathbb{E}_t\left[u\left(\left(r_0(x) + \sum_{i=1}^{n} \bar{\theta}_i (r_i(x, X_{t+1}, Y_{t+1}) - r_0(x))\right)\Phi(X_{t+1})\right) | X_t = x\right]\right)$$

Portfolio Selection with Narrow Framing (Cont'd)

Theorem

Assume for each feasible strategy, the condition that ensures the existence and uniqueness of the total utility process holds. Then,

(i) *The fixed point of* $\mathbb{W}$ *in* $\mathcal{X}_{++}$ *uniquely exists.*

(ii) *For any* $\Phi \in \mathcal{X}_{++}$ *such that* $\mathbb{W}\Phi$ *is well defined,* $\{\mathbb{W}^n\Phi\}_{n\geq 0}$ *converges to the fixed point of* $\mathbb{W}$ *in* $\mathcal{X}_{++}$.

(iii) *The fixed point* Φ^* *and the maximizer* $(c^*(x), \boldsymbol{\theta}^*(x)), x \in \mathbb{X}$ *in the dynamic programming equation are the optimal utility per unit wealth and the optimal consumption/investment strategy, respectively.*

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- We considered a portfolio selection problem with narrow framing and solved it by proving that the corresponding dynamic programming equation has a unique solution

References I

Barberis, N. and Huang, M. (2008). The loss aversion/narrow framing approach to the equity premium puzzle, *in* R. Mehra (ed.), *Handbook of the Equity Risk Premium*, Elsevier, North Holland, Amsterdam, pp. 199–236.

Barberis, N. and Huang, M. (2009). Preferences with frames: A new utility specification that allows for the framing of risks, *J. Econ. Dynam. Control* **33**(8): 1555–1576.

Barberis, N., Huang, M. and Thaler, R. H. (2006). Individual preferences, monetary gambles, and stock market participation: A case for narrow framing, *Amer. Econ. Rev.* **96**(4): 1069–1090.

De Giorgi, E. G. and Legg, S. (2012). Dynamic portfolio choice and asset pricing with narrow framing and probability weighting, *J. Econ. Dynam. Control* **36**(7): 951–972.

Epstein, L. G. and Zin, S. E. (1989). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework, *Econometrica* **57**(4): 937–969.

References II

Guo, J. and He, X. D. (2017). A new preference model that allows for narrow framing. SSRN: 2903619.

Hansen, L. P. and Scheinkman, J. A. (2012). Recursive utility in a markov environment with stochastic growth, *Proceedings of the National Academy of Sciences* **109**(30): 11967–11972.

He, X. D. and Zhou, X. Y. (2014). Myopic loss aversion, reference point, and money illusion, *Quant. Finance* **14**(9): 1541–1554.

Kreps, D. M. and Porteus, E. L. (1978). Temporal resolution of uncertainty and dynamic choice theory, *Econometrica* **46**(1): 185–200.

Ma, C. (1993). Market equilibrium with heterogeneous recursive-utility-maximizing agents, *Econ. Theory* **3**(2): 243–266.

Ma, C. (1996). Market equilibrium with heterogeneous recursive-utility-maximizing agents, corrigendum, *Econ. Theory* **7**(3): 567–570.

Ma, C. (1998). Attitudes toward the timing of resolution of uncertainty and the existence of recursive utility, *J. Econ. Dynam. Control* **23**(1): 97–112.

References III

Marinacci, M. and Montrucchio, L. (2010). Unique solutions for stochastic recursive utilities, *J. Econ.* **145**(5): 1776–1804.

Ozaki, H. and Streufert, P. A. (1996). Dynamic programming for non-additive stochastic objectives, *J. Math. Econ.* **25**(4): 391–442.