Optimal Consumption under Non-addictive Habit Formation in Incomplete Markets

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Outline

- Introduction and Literature Review
- Mathematical Model and Functional Set-up
- Main Results
- Some Future Work

Consumption Habit Formation

- The standard Merton optimal consumption problem:

$$u(x) = \sup_{H,c\in\mathcal{A}} \mathbb{E}\left[\int_0^T U(t,c_t)dt\right],$$

  where $\mathcal{A}$ is the admissible set of portfolio-consumption strategies (H,c).

- However, some empirical studies argued that
  - the von Neumann-Morgenstern utilities can not reconcile the equity premium puzzles.
  - the consumer's satisfaction level and risk tolerance sometimes rely more on recent changes.
  - the smooth consumption is more beneficial than the marked increase, such as the household consumption and expenditures with commitment.
  - ...
- The utility function should not merely be defined on the consumption rate, but also on the history pattern of the whole consumption path.

Consumption Habit Formation

- The consumption habit formation preference is defined as

$$u(x) = \sup_{H,c\in\mathcal{A}} \mathbb{E}\left[\int_0^T U(t, c_t - Z(c)_t)dt\right],$$

where the accumulative process $Z(c)$ is called the *habit formation* process which satisfies the recursive equation

$$dZ(c)_t = (\delta_t c_t - \alpha_t Z(c)_t)dt, \qquad Z(c)_0 = z.$$

- Equivalently,

$$Z(c)_t = ze^{-\int_0^t \alpha_v dv} + \int_0^t \delta_s e^{-\int_s^t \alpha_v dv} c_s ds,$$

where discounting factors α_t and δ_t measure, respectively, the persistence of the initial habits level and the intensity of consumption history. In general, α and δ are assumed to be bounded optional processes.

Addictive Habits vs Non-addictive Habits

- Addictive Habit Formation: if $U : [0, T] \times (0, +\infty) \to \mathbb{R}$, i.e., it is required that $c_t \geq Z(c)_t$ at any $t \in [0, T]$.
    - Complete Market Model with Ito processes: Detemple and Zapatero (Econometrica 1991, MF 1992), Schroder and Skiadas (RFS 2002), Englezos and Karatzas (SICON 2009)
    - General Incomplet Market Models: Yu (AAP, 2015)
    - Market Models with Transaction Costs: Yu (AAP, 2017)
- Non-addictive Habit Formation: if $U : [0, T] \times (-\infty, +\infty) \to \mathbb{R}$, the consumption rate is allowed to fall below the standard of living process.
    - Complete Market Model with Ito processes: Detemple and Karatzas (JET, 2003)
    - Incomplete Market Model: None.

Market Model

- Let us consider d risky assets modelled by a d-dimensional locally bounded semimartingale $(S_t^{(1)}, \ldots, S_t^{(d)})_{t \in [0,T]}$ on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ and one riskless bond $S_t^{(0)} \equiv 1, \forall t \in [0, T]$ which is the numéraire asset.

- The self-financing wealth process $(W_t^{x,H,c})_{t \in [0,T]}$ is given by

$$W_t^{x,H,c} \triangleq x + (H \cdot S)_t - \int_0^t c_s ds, \quad t \in [0, T].$$

  The consumption policy c_t is called x-*financeable* if the no-bankruptcy condition is satisfied, i.e., $W_t^{x,H,c} \geq 0$ a.s. for $t \in [0, T]$. Let $\mathcal{A}_x$ denotes the set of x-financeable consumptions.

- $\mathcal{M}$ denotes the family of equivalent local martingale measures and $\mathcal{M} \neq \emptyset$.

Market Model

- The optional decomposition theorem implies the consumption budget constraint condition: the process $(c_t)_{t\in[0,T]}$ is x-financeable if and only if

$$\mathbb{E}\Big[\int_0^T c_t Y_t dt\Big] \leq x, \quad \forall Y_t \in \mathcal{M}.$$

- The **primal utility maximization problem** with non-addictive habit formation is defined as

$$u(x;z) \triangleq \sup_{c\in\mathcal{A}_x} \mathbb{U}(c) = \sup_{c\in\mathcal{A}_x} \mathbb{E}\Big[\int_0^T U(t, c_t - Z(c)_t)dt\Big], \quad x>0, z>0.$$

where $U : [0, T] \times (-\infty, \infty) \to \mathbb{R}$ satisfies the classical conditions.

- Although the habit formation is not addictive, the non-negative consumption constraint $c_t \geq 0$ is active.

Duality Approach with Auxiliary Processes

- The path-dependence structure and potential time inconsistency may break the standard DPP argument.
- The feedback form is not expected in incomplete market models and special structures of the optimal consumption process are almost hopeless from the stochastic control approach.
- The classic duality between consumption rate process $c \in \mathcal{A}_x$ and the martingale measure density $Y \in \mathcal{M}$ does not work in our model due to the path integral term in $c_t - Z(c)_t$.
- We shall apply the duality approach using the auxiliary processes to hide the path-dependence.

Duality Approach with Auxiliary Processes

- Step 1: Treat $\tilde{c}_t = c_t - \int_0^t \delta_s e^{-\int_s^t \alpha_v dv} c_s ds$ as the auxiliary primal process and denote $\widetilde{\mathcal{A}}_x$ as the set of all $\tilde{c}$ for $c \in \mathcal{A}_x$.

- Step 2: Construct the auxiliary dual process

$$\Gamma_t \triangleq Y_t + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s (\delta_v - \alpha_v) dv} Y_s ds \Big| \mathcal{F}_t \Big], \textit{ for each } Y \in \mathcal{M}.$$

  Denote $\widetilde{\mathcal{M}}$ the set of all Γ. We will have $\mathbb{E}\Big[\int_0^T c_t Y_t dt \Big] = \mathbb{E}\Big[\int_0^T \tilde{c}_t \Gamma_t dt \Big]$.

- There are many challenges:
  - Duality in which space?
  - What about the extra term $ze^{-\int_0^t \alpha_v dv}$?
  - The nonnegative constraint on $c_t \geq 0$ mandates the path dependent constraint

$$\tilde{c}_t \geq -\int_0^t \delta_s e^{\int_s^t (\delta_v - \alpha_v) dv} \tilde{c}_s ds.$$

Duality Approach with Auxiliary Processes

- Trade off between c and $\tilde{c}$: No duality for c and path-dependent constraint on $\tilde{c}$.

- The auxiliary primal space can be written as

$$\widetilde{\mathcal{A}}_x = \Big\{\tilde{c} \in \mathbb{L}^0 : \mathbb{E}\Big[\int_0^T \tilde{c}_t \Gamma_t dt\Big] \leq x, \quad \forall \Gamma \in \widetilde{\mathcal{M}}, \text{and with the constraint}$$

$$\tilde{c}_t \geq -\int_0^t \delta_s e^{\int_s^t (\delta_v - \alpha_v) dv} \tilde{c}_s ds \Big\}.$$

- Consider the product space $[0, T] \times \Omega$ with the finite measure $d\widetilde{\mathbb{P}} = dt \times d\mathbb{P}$.

- The dual space $\widetilde{\mathcal{M}}$ is not closed in any sense. Extend it to the weak-$*$ closure $\widetilde{\mathcal{D}}$, which is a set of bounded finitely additive measures $\widetilde{\mathbb{Q}}$ on $\mathcal{O}$.

Duality Approach with Auxiliary Processes

- For each $x > 0$, we have an equivalent characterization of $\widetilde{\mathcal{A}}_x$,

$$\widetilde{\mathcal{A}}_x = \Big\{\tilde{c} : \langle \tilde{c}, \widetilde{\mathbb{Q}} \rangle \leq x, \text{ for all } \widetilde{\mathbb{Q}} \in \widetilde{\mathcal{D}} \text{ and with the constraint}$$
$$\tilde{c}_t \geq -\int_0^t \delta_s e^{\int_s^t (\delta_v - \alpha_v) dv} \tilde{c}_s ds \Big\}.$$

- The **auxiliary primal utility maximization problem** is written as

$$\tilde{u}(x; z) \triangleq \sup_{\tilde{c} \in \widetilde{\mathcal{A}}_x} \mathbb{U}(\tilde{c}) = \sup_{\tilde{c} \in \widetilde{\mathcal{A}}_x} \mathbb{E}\Big[\int_0^T U(t, \tilde{c}_t - z\tilde{w}_t) dt\Big],$$

where we denote $\tilde{w}_t \triangleq e^{\int_0^t (-\alpha_v) dv}$ for all $t \in [0, T]$ as some shadow random endowments

Duality Approach with Auxiliary Processes

- For each fixed Lagrange multipliers $y > 0$ and $\xi \in \mathbb{L}^0_+$, the **auxiliary dual optimization problem** is defined by

$$v(y,\xi) = \inf_{\widetilde{\mathbb{Q}} \in \widetilde{\mathcal{D}}(y)} \mathbb{V}(\widetilde{\mathbb{Q}}; y, \xi),$$

where we define the functional $\mathbb{V}(\widetilde{\mathbb{Q}}; y, \xi)$ as

$$\mathbb{V}(\widetilde{\mathbb{Q}}; y, \xi) \triangleq \sup_{\tilde{c} \in \widetilde{\mathcal{A}}_x} \Big(\mathbb{U}(\tilde{c}) - \langle \tilde{c}, \widetilde{\mathbb{Q}} \rangle + \mathbb{E}\Big[\int_0^T \Big(\tilde{c}_t + \int_0^t \delta_s e^{\int_s^t (\delta_v - \alpha_v) dv} \tilde{c}_s ds \Big) \xi_t dt \Big] \Big).$$

- As a matter of fact, Fubini's theorem deduces that

$$\mathbb{E}\Big[\int_0^T \Big(\tilde{c}_t + \int_0^t \delta_s e^{\int_s^t (\delta_v - \alpha_v) dv} \tilde{c}_s ds \Big) \xi_t dt \Big] = \mathbb{E}\Big[\int_0^T \tilde{c}_t \tilde{\xi}_t dt \Big],$$

where $\tilde{\xi}_t \triangleq \xi_t + \delta_t \mathbb{E}\Big[\int_t^T e^{\int_t^s (\delta_v - \alpha_v) dv} \xi_s ds \Big| \mathcal{F}_t \Big]$ and satisfies

$$\tilde{\xi}_t \geq \delta_t \mathbb{E}\Big[\int_t^T \tilde{\xi}_s e^{\int_t^s (-\alpha_v) dv} ds \Big| \mathcal{F}_t \Big], \quad \text{a.s.} \quad \forall t \in [0, T].$$

Duality Approach with Auxiliary Processes

- The dual functional can be written explicitly as

$$\mathbb{V}(\widetilde{\mathbb{Q}}; y, \tilde{\xi}) = \mathbb{E}\Big[\int_0^T V(t, -\tilde{\xi}_t + \Gamma_t^{\widetilde{\mathbb{Q}}})dt\Big] - \mathbb{E}\Big[\int_0^T z\tilde{w}_t \Gamma_t^{\widetilde{\mathbb{Q}}} dt\Big] + \mathbb{E}\Big[\int_0^T z\tilde{w}_t \tilde{\xi}_t dt\Big],$$

  where $\Gamma^{\widetilde{\mathbb{Q}}}(t,\omega) = \frac{d\widetilde{\mathbb{Q}}^r}{d\widetilde{\mathbb{P}}}$ and $\widetilde{\mathbb{Q}} = \widetilde{\mathbb{Q}}^r + \widetilde{\mathbb{Q}}^s \in \widetilde{\mathcal{D}}(y)$.

- To build the duality between $\tilde{u}(x)$ and $v(y, \tilde{\xi})$: How to choose the stochastic Lagrange multiplier $\tilde{\xi}^*$?

- The answer depends on another two auxiliary problems: the unconstrained auxiliary primal and dual problems.

Unconstrained Auxiliary Problems

- Consider the enlarged admissible space for the auxiliary primal space $\widetilde{\mathcal{A}}_x$ where we consider all $x \in \mathbb{R}$,

$$\bar{\mathcal{A}}_x = \left\{ \bar{c} : \; \langle \bar{c}, \widetilde{\mathbb{Q}} \rangle \leq x, \; \text{for all } \widetilde{\mathbb{Q}} \in \widetilde{\mathcal{D}} \right\}, \quad x \in \mathbb{R}.$$

  The **auxiliary unconstrained primal utility maximization problem** is defined as

$$\bar{u}(x) = \sup_{\bar{c} \in \bar{\mathcal{A}}_x} \mathbb{E}\left[\int_0^T U(t, \bar{c}_t - z\tilde{w}_t) dt \right], \quad x \in \mathbb{R}, z > 0.$$

- The auxiliary dual problem is defined by

$$\bar{v}(y) = \inf_{\bar{\mathbb{Q}} \in \widetilde{\mathcal{D}}(y)} \mathbb{E}\left[\int_0^T \left(V(t, \Gamma_t^{\bar{\mathbb{Q}}}) - z\tilde{w}_t \Gamma_t^{\bar{\mathbb{Q}}} \right) dt \right],$$

Unconstrained Auxiliary Problems

- Value functions $\bar{u}(x)$ and $\bar{v}(y)$ are conjugate of each other, i.e.,

$$\bar{v}(y) = \sup_{x \in \mathbb{R}} [\bar{u}(x) - xy],$$
$$\bar{u}(x) = \inf_{y > 0} [\bar{v}(y) + xy].$$

- The unique dual optimizer $\bar{\mathbb{Q}}^*(y)$ and the unique primal optimizer $\bar{c}^*(x)$ satisfies

$$\bar{c}^*_t(x) = I(t, \Gamma_t^{\bar{\mathbb{Q}}^*}(y)) + z\tilde{w}_t, \quad \mathbb{P}\text{-a.s.} \quad \forall t \in [0, T],$$

where $x = -\bar{v}'(y)$.

- For each fixed $y > 0$ and $\tilde{\xi}$ such that $v(y, \tilde{\xi}) < \infty$, the dual problem admits the unique optimal solution $\widetilde{\mathbb{Q}}^*(y)$ such that

$$v(y, \tilde{\xi}) = \mathbb{V}(\widetilde{\mathbb{Q}}^*(y); y, \tilde{\xi}),$$

and $\widetilde{\mathbb{Q}}^*(y) = \bar{\mathbb{Q}}^*(y)$ which is independent of the choice of $\tilde{\xi}$.

Constrained Auxiliary Problems

- Choice of $\tilde{\xi}^*$ using unconstrained problems:

  Step 1: Construction of the endogenous stopping time

$$\tau^*(y) \triangleq \inf\{t \geq 0 : I(t, \Gamma_t^{\bar{\mathbb{Q}}^*}(y)) + z\tilde{w}_t \geq 0\} \wedge T.$$

  Step 2: Prove the following results: for $\tau^*(y) \leq t \leq T$,

$$I(t, \Gamma_t^{\bar{\mathbb{Q}}^*}(y)) + z\tilde{w}_t \geq - \int_{\tau^*(y)}^{t} \delta_s e^{\int_s^t (\delta_v - \alpha_v) dv} \Big(I(s, \Gamma_s^{\bar{\mathbb{Q}}^*}(y)) + z\tilde{w}_s \Big) ds,$$

Constrained Auxiliary Problems

- For each $y > 0$, we will construct the **valid** stochastic Lagrange multiplier $\tilde{\xi}_t^*(y)$ by

$$\tilde{\xi}_t^*(y) \triangleq 0, \quad \tau^*(y) \leq t \leq T,$$

  and

$$\tilde{\xi}_t^*(y) \triangleq \Gamma_t^{\bar{\mathbb{Q}}^*}(y) - U'(t, -z\tilde{w}_t), \quad 0 \leq t \leq \tau^*(y),$$

  which implies that

$$I(t, -\tilde{\xi}_t^*(y) + \Gamma_t^{\bar{\mathbb{Q}}^*}(y)) + z\tilde{w}_t = 0, \quad 0 \leq t \leq \tau^*(y).$$

- Let us define the dual value function

$$\tilde{v}(y) \triangleq v(y, \tilde{\xi}^*(y)),$$

  the conjugate duality between value functions $\tilde{u}(x)$ and $\tilde{v}(y)$ holds,

$$\tilde{u}(x) = \inf_{y>0}(\tilde{v}(y) + xy),$$
$$\tilde{v}(y) = \sup_{x>0}(\tilde{u}(x) - xy).$$

Constrained Auxiliary Problems

- For each initial wealth $x > 0$, the optimal solution $\tilde{c}_t^*(x)$ satisfies

$$\begin{aligned}\tilde{c}_t^*(x) &= I(t, -\tilde{\xi}_t^*(y) + \Gamma_t^{\widetilde{\mathbb{Q}}^*}(y)) + z\tilde{w}_t \\ &= I(t, -\tilde{\xi}_t^*(y) + \Gamma_t^{\bar{\mathbb{Q}}^*}(y)) + z\tilde{w}_t, \quad 0 \le t \le T.\end{aligned}$$

  and

$$\begin{aligned}\tilde{c}_t^*(x) &= 0, \quad 0 \le t \le \tau^*(y), \\ \tilde{c}_t^*(x) &= \bar{c}_t^*(\bar{x}), \quad \tau^*(y) \le t \le T,\end{aligned}$$

  where $y = \tilde{u}'(x)$ and $\bar{x} = -\bar{v}'(y)$ and $\bar{c}^*(\bar{x})$ is the optimal solution for the unconstrained problem starting with the initial value $\bar{x}$.

Constrained Auxiliary Problems

- For each $x > 0$, the optimal consumption $c_t^*(x)$ to the primal utility maximization problem exists and is unique and

$$c_t^*(x) = \tilde{c}_t^*(x) + \int_0^t \delta_s e^{\int_s^t (\delta_v - \alpha_v) dv} \tilde{c}_s^*(x) ds, \quad 0 \leq t \leq T.$$

In particular,

$$c_t^*(x) = 0, \quad 0 \leq t \leq \tau^*(y),$$
$$c_t^*(x) = \bar{c}_t^*(\bar{x}) + \int_{\tau^*(y)}^t \delta_s e^{\int_s^t (\delta_v - \alpha_v) dv} \bar{c}_s^*(\bar{x}) ds, \quad \tau^*(y) \leq t \leq T,$$

where $y = \tilde{u}'(x)$ and $\bar{x} = -\bar{v}'(y)$ and $\bar{c}^*(\bar{x})$ is the optimal solution for the unconstrained auxiliary problem.

Constrained Auxiliary Problems

- For the unconstrained primal optimizer $\bar{c}^*$, let us go back to the original market model and

$$\hat{c}_t^* \triangleq \bar{c}_t^* + \int_0^t \delta_s e^{\int_s^t (\delta_v - \alpha_v) dv} \bar{c}_s^* ds,$$

corresponds to the unconstrained optimal consumption process.

- For each $x > 0$, the optimal consumption process has the special structure that $c_t^*(x) = 0$ for $0 \le t \le \tau^*(y)$ and

$$c_t^*(x) = \hat{c}_t^*(\bar{x}) - \int_0^{\tau^*(y)} \delta_s e^{\int_s^t (\delta_v - \alpha_v) dv} \left(\hat{c}_s^*(\bar{x}) - \int_0^s \delta_u e^{-\int_u^s \alpha_v dv} \hat{c}_u^*(\bar{x}) du \right) ds$$

for $\tau^*(y) \le t \le T$ where $y = \tilde{u}'(x)$ and $\bar{x} = -\bar{v}'(y)$.

Future Work

- More explicit structures on the optimal consumption in concrete incomplete market models.
- Other types of non-addictive habit formation or nonlinear habit formations.
- Market Equilibrium under non-addictive habit formation (+ addictive habit formation).