# Multi-Martingale Optimal Transport

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# Martingale Optimal Transport (MOT) Problem in One dimension

- $(\Omega, \mathcal{F}, \mathbb{P})$ : probability space
- $X : \Omega \to \mathbb{R}$, $Y : \Omega \to \mathbb{R}$ : random variables
- cost function $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$
- (marginal constraint) $\text{Law}(X) = \mu$, $\text{Law}(Y) = \nu$
- $E(Y|X) = X$.

Study the one-step martingales (stocks) $(X, Y)$ with prescribed marginals, which **minimize** the expected cost (option price)

$$\min_{X \sim \mu, Y \sim \nu, E(Y|X) = X} \mathbb{E}_\pi c(X, Y).$$

Motivation:

- [Model-free Finance] find the minimum price of option $c(x, y)$ given market information $\mu, \nu$, that is, given the prices of call / put options.

# A structure result in dimension one

- Denote $\pi = \mathrm{Law}(X, Y)$
- Let $(\pi_x)_x$ be a disintegration (=conditional probability) of $\pi$ w.r.t. $\mu$
- $\pi(dx \cdot dy) = \pi_x(dy) \cdot \mu(dx)$

**Theorem [Hobson-Neuberger-Klimmek, Beiglböck-Juillet '13]**

**Assume:**

- $c(x, y) = \pm|x - y|$
- $\mu << \mathcal{L}^1$

**Then:** for $\mu$ - a.e. $x$,

- $\pi_x$ is concentrated at two boundary points of an interval, i.e. $\pi_x = \lambda\delta_{y^-(x)} + (1 - \lambda)\delta_{y^+(x)}$.

**Question:** What is a right generalization of this theorem in higher dimension?

# Multi-Martingale Optimal Transport (MMOT) Problem [L. '16]

- $(\Omega, \mathcal{F}, \mathbb{P})$ : probability space
- $X_i : \Omega \to \mathbb{R}$, $Y_i : \Omega \to \mathbb{R}$ : random variables, $i = 1, 2, ..., d$
- cost function $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$
- (marginal constraint) $\mathrm{Law}(X_i) = \mu_i$, $\mathrm{Law}(Y_i) = \nu_i$
- $E(Y|X) = X$, where $X = (X_1, ..., X_d)$, $Y = (Y_1, ..., Y_d)$

Study the one-step martingales (stocks) $(X, Y)$ with prescribed marginals, which **minimize** the expected cost (option price)

$$\min_{X_i \sim \mu_i, Y_i \sim \nu_i, E(Y|X) = X} \mathbb{E}_\pi c(X, Y).$$

Motivation:

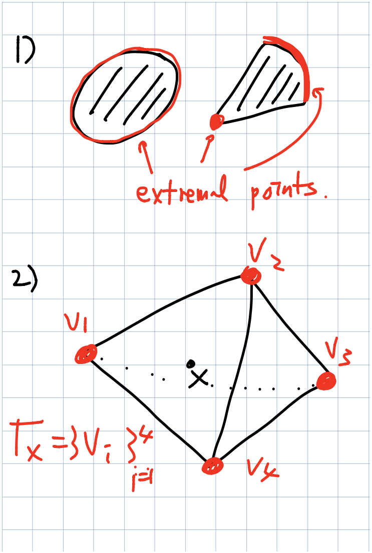
- [Finance] find the minimum price of the option whose value depends on many stocks $(X_i, Y_i)$, $i = 1, ..., d$, given the information that can be observed from the market.

# Extremal structure of MMOT in general dimension

**Theorem [L. '16]**

**Assume:**

- $\mu_i \leq_c \nu_i$ (convex order)
- $\mu_i << \mathcal{L}^1$
- $c(x, y) = \pm||x - y||$ where $||\cdot||$ is any strictly convex norm on $\mathbb{R}^d$
- $\pi = \mathrm{Law}(X, Y) \in P(\mathbb{R}^{2d})$ is any minimizer of MMOT
- $\pi(dx \cdot dy) = \pi_x(dy) \cdot \pi^1(dx)$, where $\pi^1 := \mathrm{Law}(X) \in P(\mathbb{R}^d)$.




**Then:** support of $\pi_x$ consists of the extreme points of a convex set:

$$\mathrm{supp}\,\pi_x = \mathrm{Ext}\left(\mathrm{conv}(\mathrm{supp}\,\pi_x)\right), \quad \pi^1 - a.e.\ x.$$

# How to obtain such structure result? Study the Dual Optimizer of MOT

- We say that a triple of functions $(\phi, \psi, h)$ is a dual maximizer of the MOT problem, if for every minimizer $\pi$ of MOT we have

$$\phi(x) + \psi(y) + h(x) \cdot (y - x) \leq c(x, y) \quad \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \tag{1}$$

$$\phi(x) + \psi(y) + h(x) \cdot (y - x) = c(x, y) \qquad \pi - a.e.\,(x, y). \tag{2}$$

- $\phi(x) + \psi(y) + h(x) \cdot (y - x)$ can be interpreted as an optimal subhedging strategy for the option $c(x, y)$.

# Irreducibility of $(\mu, \nu)$ is essential to achieve duality in MOT

- Beiglböck-Juillet, Beiglböck-Nutz-Touzi showed that in dimension one ($d = 1$), duality is attained if the marginals $(\mu, \nu)$ are **irreducible**.

- The irreducibility of $(\mu, \nu)$ is characterized by their *potential functions*

$$u_\mu(x) := \int |x - y| \, d\mu(y), \quad u_\nu(x) := \int |x - y| \, d\nu(y).$$

- This is also where the OT and MOT are divergent: in OT theory essentially no relation between $\mu, \nu$ is required for duality.

- The seemingly harmless linear term $h(x) \cdot (y - x)$ drastically changes the picture.

# Duality in MMOT (is also attained!)

**Theorem [L. '16]** **Assume:**

- $(\mu_i, \nu_i)$ is irreducible, $\forall i = 1, ..., d$
- $\pi$ is any minimizer of MMOT

**Then:** there exist a bunch of functions $\phi_i, \psi_i : \mathbb{R} \to \mathbb{R}$, i=1,...,d, $h : \mathbb{R}^d \to \mathbb{R}^d$ which is a dual maximizer:

$$\sum_{i=1}^{d} \phi_i(x_i) + \sum_{i=1}^{d} \psi_i(y_i) + h(x) \cdot (y - x) \le c(x, y) \quad \forall x \in \mathbb{R}^d, \forall y \in \mathbb{R}^d,$$

$$\sum_{i=1}^{d} \phi_i(x_i) + \sum_{i=1}^{d} \psi(y_i) + h(x) \cdot (y - x) = c(x, y) \quad \pi - a.e. \, (x, y).$$

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- But not only this, we find that $\mathrm{Law}(X)$ and $\mathrm{Law}(Y)$ also solve a classical dual optimal transport problem:

## $\mathrm{Law}(X)$, $\mathrm{Law}(Y)$ are also optimizers for OT

**Theorem [L. '16]** **Assume:**

- $(\phi_i, \psi_i, h_i)_{i \le d}$ is a dual maximizer
- $\pi = \mathrm{Law}(X, Y)$ is any minimizer of MMOT

**Then:** its induced *d*-copulas $\pi^1, \pi^2$ (i.e. $\pi^1 = \mathrm{Law}(X)$, $\pi^2 = \mathrm{Law}(Y)$) solve the dual optimal transport problem with respect to the costs $\alpha$, $\beta$ respectively:

$$\sum_i \phi_i(x_i) \le \alpha(x) \quad \mu_i - a.e.\, x_i, \text{ and } \sum_i \phi_i(x_i) = \alpha(x) \quad \pi^1 - a.e.\, x,$$

$$\sum_i \psi_i(y_i) \ge \beta(y) \quad \nu_i - a.e.\, y_i, \text{ and } \sum_i \psi_i(y_i) = \beta(y) \quad \pi^2 - a.e.\, y.$$

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- Here the functions $\alpha : \mathbb{R}^d \to \mathbb{R}$, $\beta : \mathbb{R}^d \to \mathbb{R}$ are naturally defined in terms of the function $y \mapsto \sum_{i=1}^d \psi_i(y_i)$ and are called the martingale Legendre transform. (**Ghoussoub-Kim-L. '15**)

- OT theory can enter for the study of the structure of $\pi^1, \pi^2$.

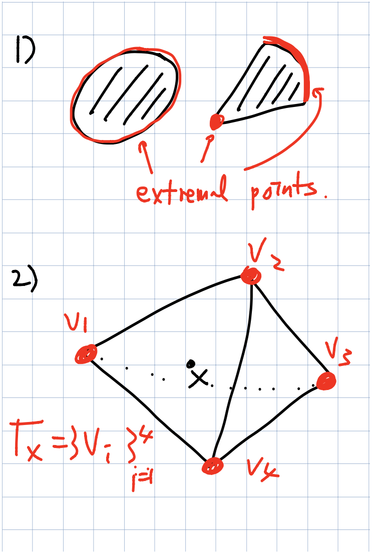
# Extremal structure of MMOT: norm-type option

**Theorem [L. ’16]**

**Assume:**

- $\mu_i \leq_c \nu_i$ (convex order)
- $\mu_i << \mathcal{L}^1$
- $c(x, y) = \pm||x - y||$ where $||\cdot||$ is any strictly convex norm on $\mathbb{R}^d$
- $\pi = \mathrm{Law}(X, Y) \in P(\mathbb{R}^{2d})$ is any minimizer of MMOT
- $\pi(dx \cdot dy) = \pi_x(dy) \cdot \pi^1(dx)$, where $\pi^1 := \mathrm{Law}(X) \in P(\mathbb{R}^d)$.

**Then:** support of $\pi_x$ consists of the extreme points of a convex set:




$$\mathrm{supp}\,\pi_x = \mathrm{Ext}\left(\mathrm{conv}(\mathrm{supp}\,\pi_x)\right), \quad \pi^1 - a.e.\ x.$$

# Extremal structure of MMOT: maximum covariance

**Example [L. '16]** **Assume:**

- $d = 2$ & dual maximizer exists ( e.g. $(\mu_i, \nu_i)$ is irreducible)
- $c(x, y) = -y_1 y_2$ or equivalently $c(x, y) = \frac{1}{2}|y_1 - y_2|^2$
- $\pi = \mathrm{Law}(X, Y) \in P(\mathbb{R}^4)$ is any minimizer of MMOT
- $\pi(dx \cdot dy) = \pi_x(dy) \cdot \pi^1(dx)$, where $\pi^1 := \mathrm{Law}(X) \in P(\mathbb{R}^2)$.

**Then:** there exists an increasing function $\psi : \mathbb{R} \to \mathbb{R}$ and a function $h : \mathbb{R}^2 \to \mathbb{R}$ such that $Y_2 = \psi(Y_1) + h(X)$.

**And:** $\pi^1 = \mathrm{Law}(X)$ is the quantile coupling of $\mu_1, \mu_2$ (that is, $\pi^1$ must be supported on the graph of an increasing function).

## Conclusion:

- The duality attainment results presented so far shall serve as the cornerstones for further development of the MOT / MMOT theory, as it did so in the classical OT theory.
- As the classical optimal transport theory (in higher dimensions) has made important contributions to many areas of mathematics and economics, I believe that this new development of probabilistic optimal embedding theory in higher dimensions will have far-reaching consequences as well.

**Thank You Very Much!**