A Category of Probability Spaces and Monetary Value Measures[1]

Takanori Adachi and Yoshihiro Ryu

Research Organization of Social Sciences
Ritsumeikan University

April 24, 2017
Asian Quantitative Finance Conference, 2017
Korea Science and Technology Center, Seoul, South Korea

[1] This work was supported by JSPS KAKENHI Grant Number 26330026.

Category of Probability Spaces

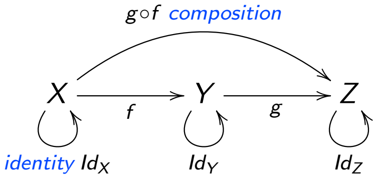
Conditional Expectation Functor

Monetary Value Measures

Concluding Remarks

Category of Probability Spaces

Some Categories that Probabilists already have

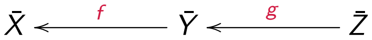


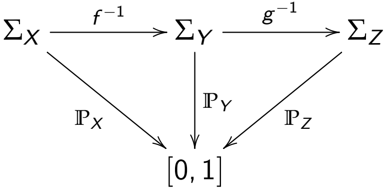
1. the category of measurable spaces (*objects* are measurable spaces and *arrows* are measurable maps)
2. a category of probability spaces (*objects* are probability spaces and *arrows* are measure-preserving maps)

We propose another category of probability spaces in this talk.

Absolutely Continuous Arrows

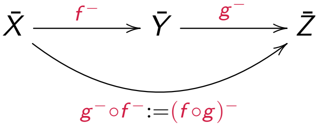
Let
$\bar{X} := (X, \Sigma_X, \mathbb{P}_X)$, $\bar{Y} := (Y, \Sigma_Y, \mathbb{P}_Y)$, $\bar{Z} := (Z, \Sigma_Z, \mathbb{P}_Z)$.





Then,
$\mathbb{P}_Y \circ f^{-1} \ll \mathbb{P}_X$ and $\mathbb{P}_Z \circ g^{-1} \ll \mathbb{P}_Y \Rightarrow \mathbb{P}_Z \circ (f \circ g)^{-1} \ll \mathbb{P}_X$.

Define arrows



Category **Prob**

Definition (Category **Prob**)

A category **Prob** is the category whose objects are all probability spaces and the set of arrows between them are defined by

$$\mathbf{Prob}(\bar{X}, \bar{Y}) \\ := \{f^- \mid f : \bar{Y} \to \bar{X} : \text{measurable with } \mathbb{P}_Y \circ f^{-1} \ll \mathbb{P}_X\}.$$

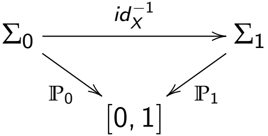
Prob is a generalization of the category χ introduced in [Adachi, 2014].

An Example of **Prob**-arrows

Suppose that we have a **Prob**-arrow like the following:

$$(X, \Sigma_0, \mathbb{P}_0) \xrightarrow{id_X^-} (X, \Sigma_1, \mathbb{P}_1)\ .$$

Then,



and we can reduce

1. $\Sigma_0 \subset \Sigma_1$
   (the information is growing),
2. $\mathbb{P}_1 = \mathbb{P}_1 \circ id_X^{-1} \ll \mathbb{P}_0$
   (the support of the probability measure is decreasing).

A **Prob***-arrow* f^- *can be considered to represent an evolving direction of information with a way of its interpretation.*

The information is evolving along f^- *but with a restriction to its accompanying probability measure.*

Category **mpProb**

Definition (Category **mpProb**)

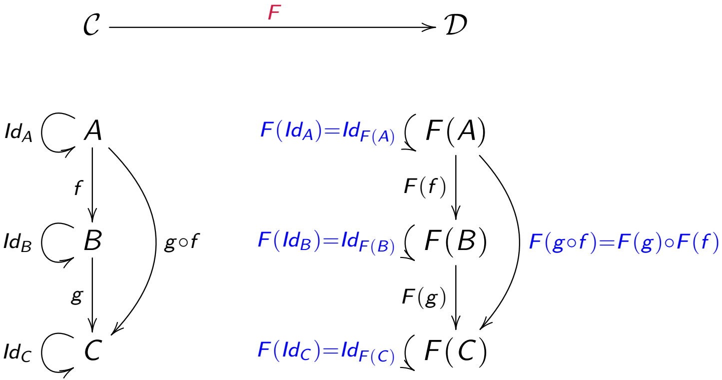
A category **mpProb** is the category whose objects are all probability spaces and the set of arrows between them are defined by

$$\mathbf{mpProb}(\bar{X}, \bar{Y}) := \{f^- \mid f : \bar{Y} \to \bar{X} : \text{measurable with } \mathbb{P}_Y \circ f^{-1} = \mathbb{P}_X\}.$$

In other words, **mpProb** is a subcategory of **Prob** whose arrows are measure-preserving.

Conditional Expectation Functor

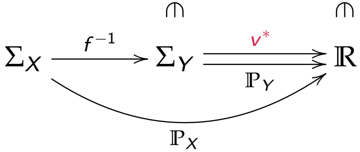
Functor $F : \mathcal{C} \to \mathcal{D}$



Conditional Expectation along f^-

Let $f^- : \bar{X} \to \bar{Y}$ be a **Prob**-arrow, and $v \in \mathcal{L}^1(\bar{Y})$.

$$B \longmapsto v^*(B) \;:=\; \int_B v \, d\mathbb{P}_Y$$



Then,

$$v^* \circ f^{-1} \ll \mathbb{P}_Y \circ f^{-1} \ll \mathbb{P}_X .$$

So we get a Radon-Nikodym derivative

$$E^{f^-}(v) := \partial(v^* \circ f^{-1})/\partial\mathbb{P}_X .$$

Then, for all $A \in \Sigma_X$

$$\int_A E^{f^-}(v) \, d\mathbb{P}_X = \int_A d(v^* \circ f^{-1})$$

$$= (v^* \circ f^{-1})(A) = v^*(f^{-1}(A)) = \int_{f^{-1}(A)} v \, d\mathbb{P}_Y .$$

Functor $\mathcal{E}$

Theorem (Conditional Expectation along $f^- : \bar{X} \to \bar{Y}$)

For all $v \in \mathcal{L}^1(\bar{Y})$ and $A \in \Sigma_X$,

$$\int_A E^{f^-}(v)\, d\mathbb{P}_X = \int_{f^{-1}(A)} v\, d\mathbb{P}_Y.$$

Theorem (Functor $\mathcal{E}$)

There exists a functor $\mathcal{E}$: $\mathbf{Prob}^{op} \to \mathbf{Set}$ as following:

$$\begin{array}{ccccccccc}
X & & \bar{X} & \overset{\mathcal{E}}{\longmapsto} & \mathcal{E}\bar{X} & := & L^1(\bar{X}) & \ni & [E^{f^-}(v)]_{\sim_{\mathbb{P}_X}} \\
f \uparrow & & f^- \downarrow & & \uparrow \mathcal{E}f^- & & & & \uparrow \mathcal{E}f^- \\
Y & & \bar{Y} & \overset{\mathcal{E}}{\longmapsto} & \mathcal{E}\bar{Y} & := & L^1(\bar{Y}) & \ni & [v]_{\sim_{\mathbb{P}_Y}}.
\end{array}$$

We call $\mathcal{E}$ a conditional expectation functor.

Unconditional Expectation

Proposition (Initial Object)

A probability space $\mathbb{I} := (\{*\}, \{\{*\}, \emptyset\}, \mathbb{P}_{\mathbb{I}})$, *where* $\mathbb{P}_{\mathbb{I}}(\{*\}) := 1$ *and* $\mathbb{P}_{\mathbb{I}}(\emptyset) := 0$, *is an initial object of the category* **Prob**.

Proposition (Unconditional Expectation)

Let $!_{\bar{Y}} : \mathbb{I} \to \bar{Y}$ *be a unique* **Prob**-*arrow and* $v \in \mathcal{L}^1(\bar{Y})$. *Then, we have*

$$E^{!_{\bar{Y}}}(v)(*) = \mathbb{E}^{\mathbb{P}_Y}[v].$$

Proof.

$$E^{!_{\bar{Y}}}(v)(*) = \int_{\{*\}} E^{!_{\bar{Y}}}(v) \, d\mathbb{P}_{\mathbb{I}} = \int_{!_Y^{-1}(\{*\})} v \, d\mathbb{P}_Y = \int_Y v \, d\mathbb{P}_Y = \mathbb{E}^{\mathbb{P}_Y}[v].$$

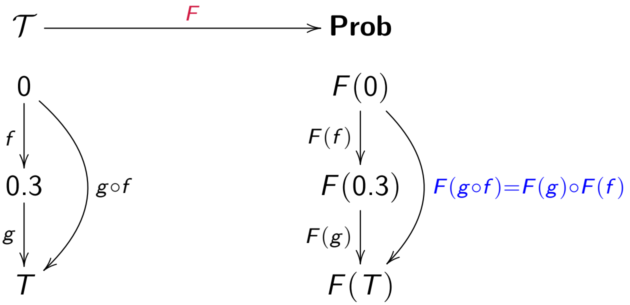
□

Filtrations are Functors

Let $\mathcal{T}$ be a fixed small category which we sometimes call the *time domain*.

Definition

A *$\mathcal{T}$-filtration* is a functor $F : \mathcal{T} \to \mathbf{Prob}$.



F-adapted processes, F-maritngales, ...

Monetary Value Measures

Dynamic Monetary Value Measures

Definition

For a σ-field $\mathcal{G} \subset \mathcal{F}$, $L(\mathcal{G}) := L^\infty(\Omega, \mathcal{G}, \mathbb{P}|\mathcal{G})$.

Definition

Let $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ be a filtration.
A *dynamic monetary value measure* (*dynamic MVM*) is a collection of functions

$$\varphi = \{\varphi_t : L(\mathcal{F}_T) \to L(\mathcal{F}_t)\}_{t\in[0,T]}$$

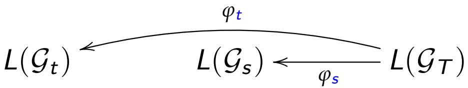
satisfying

1. *Cash invariance:*
   $(\forall X \in L(\mathcal{F}_T))(\forall Z \in L(\mathcal{F}_t))\ \varphi_t(X + Z) = \varphi_t(X) + Z$,
2. *Monotonicity:*
   $(\forall X \in L(\mathcal{F}_T))(\forall Y \in L(\mathcal{F}_T))\ X \le Y \ \Rightarrow\ \varphi_t(X) \le \varphi_t(Y)$,
3. *Normalization:* $\varphi_t(0) = 0$.

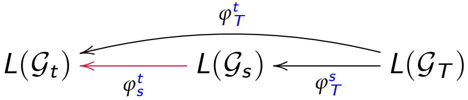
Extending Dynamic Situations

$$\varphi = \{\varphi_t : L(\mathcal{G}_T) \to L(\mathcal{G}_t)\}_{t \in [0,T]}$$

Assume $t < s < T$.



Try to extend this to like



satisfying

$$\varphi_T^t = \varphi_s^t \circ \varphi_T^s.$$

This will allow us to investigate *relative* value functions, instead of restricting their origin to the horizon T.

Monetary Value Measures (1)

A *monetary value measure* is a contravariant functor

$$\Phi : \mathbf{Prob}^{op} \to \mathbf{Set}$$

defined by

$$\begin{array}{ccccccccc}
X & & \bar{X} & \overset{\Phi}{\longmapsto} & \Phi\bar{X} & := & L^1(\bar{X}) & \ni & [\varphi^{f^-}(v)]_{\sim_{\mathbb{P}_X}} \\
{\scriptstyle f}\uparrow & & {\scriptstyle f^-}\downarrow & & \uparrow{\scriptstyle \Phi f^-} & & & & \uparrow{\scriptstyle \Phi f^-} \\
Y & & \bar{Y} & \overset{\Phi}{\longmapsto} & \Phi\bar{Y} & := & L^1(\bar{Y}) & \ni & [v]_{\sim_{\mathbb{P}_Y}}
\end{array}$$

Monetary Value Measures (2)

where φ^{f^-} satisfies

1. *Cash invariance:* $(\forall v \in \mathcal{L}^\infty(\bar{Y}))(\forall u \in \mathcal{L}^\infty(\bar{X}))$
   $\varphi^{f^-}(v + u \circ f) \sim_{\mathbb{P}_X} \varphi^{f^-}(v) + u$,
2. *Monotonicity:* $(\forall v_1 \in \mathcal{L}^\infty(\bar{Y}))(\forall v_2 \in \mathcal{L}^\infty(\bar{Y}))$
   $v_1 \lesssim_{\mathbb{P}_Y} v_2 \;\Rightarrow\; \varphi^{f^-}(v_1) \lesssim_{\mathbb{P}_Y} \varphi^{f^-}(v_2)$,
3. *Normalization:* $\varphi^{f^-}(0_Y) \sim_{\mathbb{P}_X} 0_X$
   if f^- is measure-preserving,
4. $v \in \mathcal{L}^\infty(\bar{Y})$ implies $\varphi^{f^-}(v) \in \mathcal{L}^\infty(\bar{X})$
   if f^- is measure-preserving.

We sometimes write $\Phi[\varphi^\cdot]$ for Φ for explicitly noting that arrows mapped by Φ are determined by $\varphi^\cdot$.

Entropic Value Measures

Proposition (Entropic Value Measures)

Let $f^- : \bar{X} \to \bar{Y}$ be a **Prob***-arrow, and λ be a positive real number. Define a function $\varphi^{f^-} : L^1(\bar{Y}) \to L^1(\bar{X})$ by*

$$\varphi^{f^-}(v) := \lambda^{-1} \log E^{f^-}(e^{\lambda v}), \ (\forall v \in \mathcal{L}^1(\bar{Y})).$$

Then, $\Phi := \Phi[\varphi^{\cdot}]$ is a monetary value measure. We call this Φ an entropic value measure.

Properties of MVM

Theorem

Let $\Phi = \Phi[\varphi^{\cdot}] : \mathbf{Prob}^{op} \to \mathbf{Set}$ *be a monetary value measure, and* $\bar{X} \xrightarrow{f^-} \bar{Y} \xrightarrow{g^-} \bar{Z}$ *be arrows in* **Prob**.

1. *If* f^- *is measure-preserving, we have* $\Phi f^- \circ Lf^- = Id_{L\bar{X}}$.
2. *Idempotence: If* f^- *is measure-preserving, we have* $\Phi f^- \circ Lf^- \circ \Phi f^- = \Phi f^-$.
3. *Local property:* $(\forall v_1 \in \mathcal{L}^\infty(\bar{Y}))(\forall v_2 \in \mathcal{L}^\infty(\bar{Y}))(\forall A \in \Sigma_X)$ $\Phi f^-[1_{f^{-1}(A)} v_1 + 1_{f^{-1}(A^c)} v_2]_{\sim_{\mathbb{P}_Y}} = [1_{f^{-1}(A)}]_{\sim_{\mathbb{P}_X}} \Phi f[v_1]_{\sim_{\mathbb{P}_Y}} + [1_{f^{-1}(A^c)}]_{\sim_{\mathbb{P}_X}} \Phi f[v_2]_{\sim_{\mathbb{P}_Y}}$.
4. *Dynamic programming principle: If* g^- *is measure-preserving,* $\varphi^{g^-\circ f^-}(w) = \varphi^{g^-\circ f^-}(\varphi^{g^-}(w) \circ g)$ *for* $w \in \mathcal{L}^\infty(\bar{Z})$.
5. *Time consistency:* $(\forall w_1 \in \mathcal{L}^\infty(\bar{Z}))(\forall w_2 \in \mathcal{L}^\infty(\bar{Z}))$ $\varphi^{g^-}(w_1) \lesssim_{\mathbb{P}_Y} \varphi^{g^-}(w_2) \;\Rightarrow\; \varphi^{g^-\circ f^-}(w_1) \lesssim_{\mathbb{P}_X} \varphi^{g^-\circ f^-}(w_2)$.

Concluding Remarks

Concluding Remarks

1. We introduced a category **Prob** of probability spaces.
2. A **Prob**-arrow can be considered as an evolving direction of information with a way of its interpretation.
3. We showed the existence of a conditional expectation $E^{f^-}(v)$ of a random variable v *along* a **Prob**-arrow f^-.
4. We showed that the conditional expectations form a contravariant functor $\mathcal{E}$ from **Prob** to **Set**.
5. We introduced a concept of monetary value measures as a contravariant functor from **Prob** to **Set** with a minimal axioms, which is a generalization of classical dynamic monetary value measures.
6. We showed that the monetary value measures have some favored properties such as time consistency.

References

Adachi, T. (2014).
Toward categorical risk measure theory.
Theory and Applications of Categories, 29(14):389–405.

Adachi, T. and Ryu, Y. (2016).
A category of probability spaces.
arXiv:1611.03630 [math.PR].

Adachi, T. and Ryu, Y. (2017).
Monetary value measures in a category of probability spaces.
arXiv:1702.01175 [q-fin.MF].

Detlefsen, K. and Scandolo, G. (2006).
Conditional and dynamic convex risk measures.
Working paper.