Updating Market Completion in Incomplete Market and Its Convergence

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An Overview-4

- We consider Merton consumption in a diffusion model with factor process $Y(t)$.
- The return and volatility are affected by the factor process.
- This is an incomplete market in general.
- We use the idea of market completion to complete the market.
- We consider a family of market completions indexed by θ.
- The complete market associated with θ is called θ-market.

An Overview-3

- We consider the power utility,

$$u_1(t, c) = \exp(-\rho t)\frac{1}{\gamma}c^\gamma, \; u_2(c) = \exp(-\rho T)\frac{1}{\gamma}c^\gamma.$$

- $0 < \gamma < 1$.
- The value for θ-market is given by

$$V^\theta(T, x, y) = \frac{x^\gamma}{\gamma}\exp(W^\theta(T, y)).$$

- T is the time horizon.
  x is the initial wealth.
  y is the initial state of the factor process.

An Overview-2

- We derive an updating scheme for market completion.
- At nth-iteration, we have θ_n-market with value

$$V^{\theta_n}(T, x, y) = \frac{x^\gamma}{\gamma} \exp(W^{\theta_n}(T, y)).$$

- The value of the original market is given by

$$V(T, x, y) = \frac{x^\gamma}{\gamma} \exp(W(T, y)).$$

- We obtain

$$0 \leq W^{\theta_n}(T, y) - W(T, y) \leq \lambda^n (W^{\theta_0}(T, y) - W(T, y)).$$

- This gives the exponential convergence of W^{θ_n} to W.
- We explain some results from Fleming-Nagai-Sheu (2017)

An Overview-1

- Such studies shall have some value.
- In this conference, we see several studies for complete markets, interesting results can be obtained for complete markets.
- Using the idea to be decribed in the following, the problem in incomplete markets may be solved by iteration.

Introduction-7

- The first use of martingale method for the portfolio optimization problems in complete market is very successful.
- The initial studies can be found in Pliska(1986), Karatzas-Lehoczky-Shreve(1987) and Cox-Huang(1989).
- Martingale method uses duality argument from convexity analysis and martingale representation theorem in stochastic calculus.

Introduction-6

- The first step: to study a static optimization problem with constraint.
- The duality argument from convexity analysis can be applied.

- The second step: use the martingale representation theorem to find an optimal trading strategy.

- Martingale method can be applied to general complete market models.

- It can be used when general utility functions are considered.

Introduction-5

- In an **incomplete market**, the martingale method introduces an idea: **market completion.**
- Additional ”fictitious” stocks are included in the market.
- Market becomes complete.
- Pages[1987], He-Pearson[1990], Karatzas-Lehoczky-Shreve-Xu[1991].
- In fictitiou complete market, we solve the problem by using martingale method.
- The value is larger than the value in the original market.
- **Big question**: How to reduce the gap?

Introduction-4

- This idea is used to find an approximation of solution for optimal consumption problem, a good one if the gap is small.
- B. Haugh, L. Kogan and J. Wang (2003), P. Zaczkowski and C Rogers (2013)
- Our study (Fleming-Nagai-Sheu (2017)) finds an idea to ”update” the market completion.
- This updating scheme can reduce the gap.
- Here we discuss a case that the efficiency of gap reduction is of exponential.

Introduction-3

- We consider a factor model.
- The dynamic programming approach can be applied to derive the HJB equation.
- Solving the HJB equation can be used to derive a candidate of optimal strategy.
- We can rewrite the HJB equation as Isaacs equation of inf-sup type.
- This observation is crucial in our study.

Introduction-2

- This introduces a stochastic game consideration.
- Attempting to find the saddle point of the game problem suggests an updating scheme.
- As another consequence, one shows the minimal gap is 0.

Introduction-1

- Merton (1969) considered simple model.
  Dynamic programming approach is used to solve the problem.
- More complicated models are considered in Fleming-Pang(2004), Fleming-Hernandez(2003, 2005), Hata-Sheu (2012 a, b), Hata-Nagai-Sheu (2017).

Outline

- Merton's consumption problem.
- Martingale method in complete market.
- "Fictitious" market completion in incomplete market.
- Factor model
- HJB (Hamilton-Jacobi-Bellman) equation
- HJB equation as Isaacs equation.
- Updating scheme.
- Exponential Convergence

Optimal Consumption Problem

General Utility Functions

$$(C') \qquad \sup_{(c,\pi)\in\mathcal{A}} E\left[\int_0^T e^{-\rho t} u_1(c_t X_t^{c,\pi})dt + e^{-\rho T} u_2(X_T^{c,\pi})\right]$$

- u_1, u_2 are utility functions: increasing and concave.
- $X_t^{c,\pi}$ is the wealth process with strategy (c,π).
- π_t is a trading strategy.
- $c_t X_t^{c,\pi}$ is the rate of consumption.
- $\mathcal{A}$: adimissible strategies
- The dynamics of $X_t^{c,\pi}$ and the relation with c,π will be decribed later.

Power Utility Functions

$$(C) \qquad \sup_{(c,\pi)\in\mathcal{A}} E\left[\int_0^T e^{-\rho t}\frac{1}{\gamma}(c_t X_t^{c,\pi})^\gamma dt + e^{-\rho T}\frac{1}{\gamma}(X_T^{c,\pi})^\gamma\right],$$

Complete Market-15

- Price process of stocks: $i = 1, 2, \cdots, N$,

$$dS_i(t) = S_i(t)\left\{\mu_i(t)dt + \sum_{j=1}^{n}\sigma_{ij}(t)dB_j(t)\right\}.$$

- $\mu_i(t), \sigma_{ji}(t)$ are progressively measurable with respect to $\mathcal{F}_t^B$.
- $B(t)$ is Brownian motion.
- $\mathcal{F}_t^B$ is the filtration generated by $B(\cdot)$.
- $\sigma(t) = (\sigma_{ij}(t))$.
- $r(t) \geq 0$ is the interest rate.
  $r(t)$ is progressively measurable.
- **Complete market**: $\sigma(t)$ is invertible.

Wealth Process

- Investment strategy π_t, c_t:

$$\pi(t) = (\pi_1(t), \pi_2(t), \cdots, \pi_N(t))$$

- $\pi(t)$ is the amount of wealth on asset i.
- c_t is the amount of consumption rate.
- $X^{\pi,c}(t)$ is the wealth process.
- Dynamics of $X^{\pi,c}(t)$:

$$dX^{\pi,c}(t) = \sum_{i=1}^{N} \pi_i(t) \frac{dS_i(t)}{S_i(t)} + (X^{\pi,c} - \sum_{i=1}^{N} \pi_i(t)) \frac{dS_0(t)}{S_0(t)} - c_t dt.$$

Complete Market-13

- Then

$$dX^{\pi,c} = \left\{ r \cdot X^{\pi,c} + (\sigma^T \pi)^T \theta^0 - c \right\} dt + (\sigma^T \pi)^T dB.$$

We omit the dependence on t.

- $\pi(\cdot), c(\cdot)$ is admissible if $X^{\pi,c}(t) \geq 0$ for all t.
- The market is complete, we define

$$\theta^0(t) = \sigma(t)^{-1}(\mu(t) - r(t)\mathbf{1}).$$

$\mathbf{1} = (1, 1, \cdots, 1)^T \in R^N$

- $u_1(t, c), u_2(c)$ are utility functions.
- For $u(c) = u_1(t, c)$ or $u(c) = u_2(c)$, we assume the following conditions:
  (a) $u(c)$ is concave , nondecreasing, finite for $c > 0$.
  (b) $u'(c) > 0$ is continuous, strictly decreasing and $u'(\infty) = 0$.
- The optimal consumption is to maximize

$$V(x) = \sup_{c,\pi} E[\int_0^T u_1(t, c(t))dt + u_2(X^{x,c,\pi}(T)].$$

- A solution is given as follows.

Complete Market-11

- Define conjugate functions,

$$\hat{u}_1(t,z) = \sup_c \{u_1(t,c) - cz\}, \ \hat{u}_2(z) = \sup_c \{u_2(c) - cz\}.$$

- $I_1(t,\cdot)$ is the inverse function of $u_1'(t,\cdot)$.
  $I_2(\cdot)$ is the inverse function of $u_2'(,\cdot)$.
- $\hat{u}_1(t,z) = u_1(t, I_1(t,z)) - I_1(t,z)z$,
  $\hat{u}_2(z) = u_2(I_2(z)) - I_2(z)z$.

Complete Market-10

- $Z^0(t) = \exp(-\int_0^t \theta(u)^T dB(u) - \frac{1}{2}\int_0^t |\theta(s)|^2 ds)$.
- $H^0(t) = \frac{Z^0(t)}{S^0(t)}$.
- We assume the following condition hold:

$$\mathcal{X}(z) = E\left[\int_0^T H^0(t) \cdot I_1(t, zH^0(t))dt + H^0(T) \cdot I_2(zH^0(T))\right] < \infty.$$

Theorem 1

- Assume **the market is complete** and $\mathcal{X}(z) < \infty$ for $z > 0$.
- Define $\mathcal{Z}(\cdot)$ the inverse of $\mathcal{X}(\cdot)$.
- Define $c^*(t) = I_1(t, \mathcal{Z}(x)H^0(t))$,

$$M^*(t) = E[\int_0^T H^0(t)\cdot I_1(t, \mathcal{Z}(x)H^0(t))dt + H^0(T)\cdot I_2(\mathcal{Z}(x)H^0(T))|\mathcal{F}_t],$$

- $\psi^*(t)$ is given by the martingale representation,

$$M^*(t) = x + \int_0^t \psi^*(s)^T dB(s).$$

- $X^*(t)$ is defined by the relation,

$$H^0(t)X^*(t) + \int_0^t H^0(s)c^*(s)ds = M^*(t).$$

(cont'd)

Theorem 1 (cont'd)

- $\pi^*(t)$ is defined by the relation,

$$\psi^*(t) = H^0(t)\left\{\sigma(t)^T\pi^*(t) - X^*(t)\theta^0(t)\right\}.$$

- Then $c^*(t), \pi^*(t)$ is a solution of the optimal consumption problem and

$$X^*(t) = X^{x,c^*,\pi^*}(t).$$

Complete Market-7

Main Idea

Budget constraint

$$E\left[\int_0^T H^0(t)c(t)dt + H^0(T)X^{\pi,c}(T)\right] \le \infty.$$

Static Problem with Constraint

- Optimization problem

$$\sup_{c,\xi} E\left[\int_0^T u_1(t, c(t))dt + u_2(\xi)\right]$$

- Constraint

$$E\left[\int_0^T H^0(t)c(t)dt + H^0(T)\xi\right] \le \infty.$$

Martingale Representation

Proposition 1

- Assume the market is complete.
- $c(s)$ is nonnegative progressively measurable.
- ξ is $\mathcal{F}_T$ measurable nonnegative random variable
- Assume the relation,

$$E[H^0(T)\xi + \int_0^T H^0(s)c(s)ds] = x.$$

- Then there is $\pi(\cdot)$ such that

$$\xi = X^{\pi,c}(T).$$

- Denote $M(t)$ the martingale

$$M(t) = E[H^0(T)\xi + \int_0^T H^0(s)c(s)ds|\mathcal{F}_t].$$

(cont'd)

Proposition 1 (cont'd)

- Then $X^{\pi,c}(t)$ is defined from the relation,

$$H^0(t)X^{\pi,c}(t) + \int_0^t H^0(s)c(s)ds = M(t).$$

- Let the martingale representation of $M(t)$ be given by

$$M(t) = x + \int_0^t \psi(s)dB(s).$$

- Then $\pi(t)$ is given by the relation:

$$\psi(t) = H^0(t)(\sigma(t)\pi(t) - X^{x,c,\pi}(t)\theta(t))$$

- Reference: Karatzas-Shreve.

Power Utility

- Let $\gamma < 1, \gamma \neq 0$. $\rho > 0$ be constants. We take

$$u_1(t,c) = u_\gamma(t,x),\ u_2(c) = u_\gamma(T,c),$$

where $u_\gamma(t,c) = e^{-\rho t}\frac{1}{\gamma}c^\gamma$

- Then

$$I_1(t,z) = e^{-\frac{\rho}{1-\gamma}t} z^{-\frac{1}{1-\gamma}},\quad I_2(z) = e^{-\frac{\rho}{1-\gamma}T} z^{-\frac{1}{1-\gamma}}.$$

- Conjugate functions are

$$\hat{u}_1(t,z) = -\frac{1-\gamma}{\gamma} e^{-\frac{\rho}{1-\gamma}t} z^{-\frac{\gamma}{1-\gamma}},\ \hat{u}_2(z) = -\frac{1-\gamma}{\gamma} e^{-\frac{\rho}{1-\gamma}T} z^{-\frac{\gamma}{1-\gamma}}.$$

Complete Market-3

- $\mathcal{X}(z)$ is given by

$$z^{-\frac{1}{1-\gamma}} E[\int_0^T e^{-\frac{\rho}{1-\gamma}t} H^0(t)^{-\frac{\gamma}{1-\gamma}} dt + e^{-\frac{\rho}{1-\gamma}T} H^0(T)^{-\frac{\gamma}{1-\gamma}}].$$

- $\mathcal{X}(z) < \infty$ if

$$\phi = E[\int_0^T e^{-\frac{\rho}{1-\gamma}t} H^0(t)^{-\frac{\gamma}{1-\gamma}} dt + e^{-\frac{\rho}{1-\gamma}T} H^0(T)^{-\frac{\gamma}{1-\gamma}}] < \infty.$$

- $\mathcal{Z}(x) = (\frac{x}{\phi})^{-(1-\gamma)}$.

Theorem 2

- Let $u_1(t,c) = u_\gamma(t,x),\ u_2(c) = u_\gamma(T,c), \quad \gamma < 1, \gamma \neq 0.$
- Assume

$$\phi = E[\int_0^T e^{-\frac{\rho}{1-\gamma}t} H^0(t)^{-\frac{\gamma}{1-\gamma}} dt + e^{-\frac{\rho}{1-\gamma}T} H^0(T)^{-\frac{\gamma}{1-\gamma}}] < \infty.$$

- Define

$$c^*(t) = \frac{x}{\phi} e^{-\frac{\rho}{1-\gamma}t} H^0(t)^{-\frac{1}{1-\gamma}},$$

- the martingale

$$M^*(t) = \frac{x}{\phi} E[\int_0^T e^{-\frac{\rho}{1-\gamma}s} H^0(s)^{-\frac{\gamma}{1-\gamma}} ds + e^{-\frac{\rho}{1-\gamma}T} H^0(T)^{-\frac{\gamma}{1-\gamma}} | \mathcal{F}_t]$$

- and the martingale representation

$$M^*(t) = x + \int_0^t \psi^*(t)^T dB(t). \quad \textbf{(continue)}$$

Theorem 2 (continue)

- Define

$$\phi(t) = E[\int_t^T e^{-\frac{\rho}{1-\gamma}t} H^0(t,s)^{-\frac{\gamma}{1-\gamma}} dt + e^{-\frac{\rho}{1-\gamma}T} H^0(t,T)^{-\frac{\gamma}{1-\gamma}} | \mathcal{F}_t],$$

- Here $H^0(t,s) = \frac{H^0(s)}{H^0(t)}$, $0 \leq t < s \leq T$,
- Define $X^*(t) = \frac{x}{\phi} e^{-\frac{\rho}{1-\gamma}t} H^0(t)^{-\frac{\gamma}{1-\gamma}} \phi(t)$,
- $\pi^*(t) = \sigma(t)^{-1}(\psi^*(t)\frac{1}{H^0(t)} + X^*(t)\theta^0(t))$.
- Then $X^*(t) = X^{x,c^*,\pi^*}(t)$ and $c^*(\cdot), \pi^*(\cdot)$ is a solution.
- The optmal consumption problem has value given by

$$V(x) = \frac{x^\gamma}{\gamma}\phi^{1-\gamma}.$$

Incomplete Market-5

- Price process of stocks: $i = 1, 2, \cdots, N$,

$$dS_i(t) = S_i(t)\left\{\mu_i(t)dt + \sum_{j=1}^{d}\sigma_{ij}(t)dB_j(t)\right\}.$$

  $\mu_i(t), \sigma_{ji}(t)$ are progressively measurable with respect to $\mathcal{F}_t$
  .
- Denote $\sigma(t) = (\sigma_{ij}(t))$.
- $r(t) \geq 0$ is the interest rate.
  $r(t)$ is progressively measurable.
- We assume $\sigma(t)\sigma(t)^T$ is invertible.
- $\sigma(t)$ **may not be invertible.**
  The market may not be complete.

Incomplete Market-4

- Define $\sigma_f(t) = I - \sigma(t)^T(\sigma(t)\sigma(t)^T)^{-1}\sigma(t)$.
- Projection property: $\sigma_f(t)\sigma_f(t) = \sigma_f(t)$.
- $\sigma(t)\sigma_f(t) = 0, \ \sigma_f(t)\sigma(t)^T = 0$. A
- Market price of risk:
  $\theta^0(t) = \sigma(t)^T(\sigma(t)\sigma(t)^T)^{-1}(\mu(t) - t(t)\mathbf{1})$.

Market Completion

- Let $\theta(t)$ be a progressively measurable R^d valued process.
- We consider a market which has stocks with prices $S_1(t), S_2(t), \cdots, S_N(t)$.
- We also include"fictitious" stocks with price dynamics:

$$d\tilde{S}_j^{\theta}(t) = \tilde{S}_j^{\theta}(t)\left\{(r(t) + \sigma_f(t)^{jk}\theta_j(t))dt + \sigma_f^{jk}(t)dB_k(t))\right\}.$$

- The dynamics of $S_i(t)$ can be rewritten as

$$dS_i(t) = S_i(t)\left\{(r(t) + \theta_i^0(t))dt + \sigma^{ik}(t)dB_k(t))\right\}.$$

- A trading strategy:
  $\pi_i(t)$, the amount in S_i, and $\tilde{\pi}_j(t)$, the amount in $\tilde{S}_j$.
- The amount of consumption rate is $c(t) \geq 0$.

- Dynamics of the wealth process $\overline{X}(t) = X^{x,c,\pi,\tilde{\pi}}(t)$:

$$d\overline{X}(t) = \left\{ r(t) + \bar{\pi}(t)^T(\theta^0(t) + \sigma_f(t)\theta(t)) \right\} dt + \bar{\pi}(t)^T dB(t).$$

- Here

$$\bar{\pi}(t) = \sigma(t)^T \pi(t) + \sigma_f(t)\tilde{\pi}(t),$$

$$\theta^0(t) = \sigma(t)^T(\sigma(t)\sigma(t)^T)^{-1}(\mu(t) - r(t)\mathbf{1}).$$

- One to one correspondence of $\overline{\pi}$ and $(\pi, \tilde{\pi})$:

$$\pi = (\sigma\sigma^T)^{-1}\sigma(t)\overline{\pi}, \quad \tilde{\pi} = \sigma_f\overline{\pi}.$$

Incomplete Market-1

- We consider power utility. The value for θ-market:

$$V^{\theta}(x) = \frac{x^{\gamma}}{\gamma}(\phi^{(\theta)})^{1-\gamma}.$$

- Here

$$\phi^{(\theta)} = E[\int_0^T \exp(-\frac{\rho}{1-\gamma}t)H^{\theta}(t)^{-\frac{\gamma}{1-\gamma}}dt + \exp(-\frac{\rho}{1-\gamma}T)H^{\theta}(T)^{-\frac{\gamma}{1-\gamma}}].$$

- We have $V(x) \leq V^{\theta}(x)$.

Factor Models-2

- Prices of risky assets: $S_i(t)$, $i = 1, 2, \cdots, N$,

$$dS_i(t) = S_i(t)\left\{\mu^i(Y(t))dt + \sigma_P^{ij}(Y(t))dB_j(t)\right\},$$

- Bank account: interest rate is $r(Y(t))$.
- Factor process: $Y(t) = (y_1(t), \cdots, y_n(t))$.

$$dY(t) = b(Y(t))dt + \sigma_F(Y(t))dB(t).$$

Factor Models-1

Dynamics of Wealth Process

$$dX^{\pi,c}(t) = X^{\pi,c}(t)\{(\sum_i \pi_i(t)\bar{\mu}^i(Y(t)) + r(Y(t)) - c(t))dt + \pi_i(t)\sigma_P^{ij}(Y_t)dB_j(t)\}.$$

Dynamics of Factor Process

$$dY(t) = b(Y(t))dt + \sigma_F(Y(t))dB(t).$$

Optimization Problem:

(C)

$$V(T,x,y) = \sup_{(c,\pi)\in\mathcal{A}} E[\int_0^T e^{-\rho t}\frac{1}{\gamma}(c_t X_t^{c,\pi})^\gamma dt + e^{-\rho T}\frac{1}{\gamma}(X_T^{c,\pi})^\gamma],$$

HJB Equations-4

- Dynamic programming approach is a very useful idea used in control theory.
- HJB equation can de derived.
  HJB is a nonlinear partial differential equation.
- A solution of HJB equation gives a candidate of optimal strategy.
- Merton (1970) derives an optimal trading strategy for an investment problem.

HJB Equation-3

- **HJB equation**: by a standard argument,
  (HJB)

$$\frac{\partial V}{\partial t} = \frac{1}{2}\sum_{ij} a_F^{ij}(y)D_{ij}V + b(y)^T D_y V + \sup_{c,\pi}\left[\frac{c^\gamma x^\gamma}{\gamma} - \rho V + x\pi^T\sigma_F(y)\sigma_P(y)^T D_{xy}V + \frac{1}{2}x^2\pi^T\sigma_P(y)\sigma_P(y)^T D_{xx}V + xD_xV(x,y)\{r(y)+\pi^T\bar{\mu}(y)-c\}\right],$$

  c, π are taken $c \in [0,\infty), \pi \in R^m$.

- Assuming

$$V(T,x,y) = \frac{x^\gamma}{\gamma}e^{W(T,y)}.$$

  The equation for $W(T,y)$:

$$(HJB') \quad \frac{\partial W}{\partial t} = \frac{1}{2}\sum_{ij} a_F^{ij}(y)D_{ij}W + H_\gamma(y, W(y), DW(y)).$$

HJB Equation-2

Notations:

$$H_\gamma(y,w,p) = \hat{H}_\gamma(y,p) + (1-\gamma)\exp(-\frac{w}{1-\gamma}) - \rho.$$

$$\hat{H}_\gamma(y,p) = \frac{1}{2}\sum_{ij=1}^{m} a_\gamma^{ij}(y)p_i p_j + \sum_{i=1}^{m} b_\gamma^i(y)p_i + U_\gamma(y),$$

$$a_F(y) = \sigma_F(y)\sigma_F(y)^T, \quad a_P(y) = \sigma_P(y)\sigma_P(y)^T.$$

$$b_\gamma(y) = b(y) + \frac{\gamma}{1-\gamma}\sigma_F(y)\sigma_P(y)^T a_P(y)^{-1}\bar{\mu}(y),$$

$$a_\gamma(y) = a_F(y) + \frac{\gamma}{1-\gamma}\sigma_F(y)\sigma_P(y)^T a_P(y)^{-1}\sigma_P(y)\sigma_F(y)^T,$$

$$U_\gamma(y) = \frac{\gamma}{2(1-\gamma)}\bar{\mu}(y)^T\, a_P(y)^{-1}\bar{\mu}(y) + \gamma r(y)$$

Optimal Strategy

- Assume (HJB) has a solution W. A candidate of optimal strategy $\pi^*(t), c^*(t)$:

$$\pi^*(t) = h^*(T-t, Y(t)),\ c^*(t) = c^*(T-t, Y(t)),$$

- Here

$$h^*(t,y) = \frac{1}{1-\gamma} a_P^{-1} \left\{ \bar{\mu} - r\mathbf{1}_n + \sigma_P \sigma_F^T D_y W(t,y) \right\},$$

$$c^*(t,y) = \exp(-\frac{W(t,y)}{1-\gamma}),$$

- Hata-Sheu(2012a, b), Hata-Nagai-Sheu(2017).

- We consider the family of θ-markets.
- A fundamental relation is given by

$$V(T,x,y) = \inf_\theta V^\theta(T,x,y).$$

- This can be explained by the following observation.

Lemma 1.
The HJB for $V(T,x,y)$ can be rewritten as

$$\begin{aligned}\frac{\partial V}{\partial t} = \; & \frac{1}{2} a_F^{ij} D_{y_i y_j} V + b^T D_{y_i} V + rx \; D_x V \\ & + \inf_\theta \mathbf{sup}_{\bar{\pi},\mathbf{c}} \Big[u(xc) - \rho V + \frac{1}{2} |\bar{\pi}|^2 x^2 D_{xx} V + x(\sigma_F \bar{\pi})^T D_{xy} V \\ & + x(r \; + \bar{\pi}^T(\bar{\mu} - r\mathbf{1} + \sigma_f \theta) - c) D_x V \Big] .\end{aligned}$$

$\bar{\mu}(y)$ is defined by $\bar{\mu} = \sigma_P(\sigma_P \sigma_P^T)^{-1}(\mu - r\mathbf{1}) + r\mathbf{1}$.

Market Completion and Isaacs Equation-4

- This is the Isaacs equation for a stochastic differential game.
- We can also verify **Isaacs condition.** That is,

$$\inf_{\theta} \sup_{c,\pi} \{\cdots\} = \sup_{c,\pi} \inf_{\theta} \{\cdots\}.$$

Market Completion and Isaacs Equation-3

- The θ^* take inf:

$$\theta^* = -\frac{1}{D_x V}\sigma_f(y)\sigma_F(y)^T D_{xy} V.$$

- $\overline{\pi}^*$ takes sup:

$$\overline{\pi}^* = -\frac{1}{xD_{xx} V}(\sigma_F(y)^T D_{xy} V + ((\bar{\mu}(y) - r(y)\mathbf{1}) + \sigma_f(y)\theta)D_x V).$$

- Here $\sigma_f(y) = I - \sigma_P(y)^T(\sigma_P(y)\sigma_P(y)^T)^{-1}\sigma_P(y)$.

Market Completion and Isaacs Equation-2

- θ^*-market and the original market has the same HJB.
- This explain $V(T,x,y) = V^{\theta^*}(T,x,y) = \inf_\theta V^\theta(T,x,y)$.
- The formula for θ^* tell how to do updating from $V(\cdot)$.

Market Completion and Isaacs Equation-1

- For power utility function, we have

$$V(T, x, y) = \frac{x^\gamma}{\gamma} e^{W(T,y)}.$$

- Then

$$\overline{\pi}^* = \frac{1}{1-\gamma} \left\{ \sigma_F^T D_y W(t, y) + ((\bar{\mu} - r\mathbf{1}) + \sigma_f \theta) \right\},$$

$$\theta^* = -\sigma_f(y) \sigma_F(y)^T D_y W.$$

Value Function-10

- We take θ a function $(t, Y(t))$, denoted by $\theta(t, Y(t))$.
- $\theta(t, y)$ is smooth.
- $V^{(\theta)}(x) = \frac{x^\gamma}{\gamma}\phi^{(\theta)}(0, y)^{1-\gamma}$.
- Here

$$\begin{aligned}\phi^{(\theta)}(t, y) = \quad & E_{t,y}\left[\int_t^T \exp(-\tfrac{\rho}{1-\gamma}s)H^{(\theta)}(t, s)^{-\frac{\gamma}{1-\gamma}}ds\right. \\ & \left.+\exp(-\tfrac{\rho}{1-\gamma}T)H^{(\theta)}(t, T)^{-\frac{\gamma}{1-\gamma}}\right],\end{aligned}$$

where $Y(t) = y$.

Value Function-9

- $H^{(\theta)}(t,s) = \frac{H^{(\theta)}(s)}{H^{(\theta)}(t)}, \ t < s,$

$$H^{(\theta)}(t) = \exp\left(-\int_0^t \overline{\theta}(u, Y(u))dB(u) - \frac{1}{2}\int_0^t |\overline{\theta}(u, Y(u))|^2 du\right)$$
$$\exp(-\int_0^t r(Y(u))du).$$

- $\overline{\theta}(t,y) = \theta^0(y) + \sigma_f(y)\theta(t,y)$

Value Function-8

- $W^{(\theta)}(t, y) = (1 - \gamma) \log \phi^{(\theta)}(0, y)$.
- Define $\theta'(t, y) = -\sigma_f(y)\sigma_F(y)^T DW^{(\theta)}(t, y)$.
- We have

$$V^{(\theta')}(x) = \frac{x^\gamma}{\gamma}\phi^{(\theta')}(0, y)^{1-\gamma} = \frac{x^\gamma}{\gamma}\exp(W^\theta(0, y)).$$

- Here

$$\begin{aligned}\phi^{(\theta')}(t, y) = \quad & E_{t,y}[\textstyle\int_t^T \exp(-\frac{\rho}{1-\gamma}(s-t))H^{(\theta')}(t, s)^{-\frac{\gamma}{1-\gamma}} ds \\ & + \exp(-\tfrac{\rho}{1-\gamma}(T-t))H^{(\theta')}(t, T)^{-\frac{\gamma}{1-\gamma}}],\end{aligned}$$

Theorem 3 $\quad \phi^{(\theta')}(t, y) \le \phi^{(\theta)}(t, y)$.

Updating Scheme

- Take $\theta_1 = 0$.
- Define

$$\theta_{n+1}(t,y) = -\sigma_f(y)\sigma_F(y)^* DW^{(\theta_n)}(t,y),$$

- Here

$$V^{(\theta_n)}(x) = \frac{x^\gamma}{\gamma}\phi^{(\theta_n)}(0,y)^{1-\gamma},$$

- $W^{(\theta_n)}(t,y) = (1-\gamma)\log\phi^{(\theta_n)}(t,y),$

Value Function-6

- $V^{(\theta_{n+1})} \leq V^{(\theta_n)}$. (**Theorem 3**)
- $V^{(\theta_n)}$ (and also $\phi^{(\theta_n)}$) converges.
- We expect the limit is the value V of the optimal consumption problem.
- We can try to show the limit is a solution of our HJB equation.
- The uniqueness of the HJB equation proves the limit must be the value of the optimal consumption problem
- These are interesting challenging mathematical problems.

Value Function-5

- The following is a different approach.
- There is a natural way to construct a strategy for the original market from a strategy of θ_n-market.
- We hope the value of this strategy is closed to the value of the θ_n-market when n is large.
- Then it must also be close to the optimal value of the original market

Value Function-4

- Let $(c^{\theta_n}, \overline{\pi}^{\theta_n})$ be the optimal strategy for the θ_n-market.
- $c^{\theta_n} = \exp(-\frac{W^{\theta_n}}{1-\gamma}), \quad \pi^{\theta_n} = \frac{1}{1-\gamma}(\sigma_F^T D_y W^{\theta_n} + (\theta_0 + \sigma_f \theta_n))$.
- $V^{\theta_n}(T, x, y) = \frac{x^\gamma}{\gamma} \exp(W^{\theta_n}(T, y))$.

Value Function-3

- Define the strategy $(c^{(n)}, \pi^{(n)})$ for the original market,

$$
\begin{aligned}
c^{(n)} &= c^{\theta_n}, \\
\pi^{(n)} &= (\sigma_p \sigma_p^T)^{-1} \sigma_P \overline{\pi}^{\theta_n} \\
&= \frac{1}{1-\gamma} (\sigma_p \sigma_p^T)^{-1} \sigma_P (\sigma_F^T D_y W^{\theta_n} + \theta_0).
\end{aligned}
$$

- The value is $J(x, y, \pi^{(n)}, c^{(n)})$
- We hope to compare the value of this strategy and $V^{(\theta_n)}$.
- Since

$$V^{(\theta_n)} \geq V \geq J(x, y, \pi^{(n)}, c^{(n)}),$$

  we may compare $V^{(\theta_n)}$ and V.

Value Function-2

Theorem 4 $0 < \gamma < 1$,

- The value of θ_n-market $V^{(\theta_n)}(0,x,y) = \frac{x^\gamma}{\gamma}\exp(W^{(\theta_n)}(0,y))$.
- The value of the original market $V(0,x,y) = \frac{x^\gamma}{\gamma}\exp(W(0,y))$.
- $(c^{(n)}, \pi^{(n)})$ is the strategy for the original market constructed above.
- $J(x,y,c^{(n)},\pi^{(n)})$ is the value of $c^{n)}, \pi^{(n)}$.
- Then

$$J(x,y,c^{(n)},\pi^{(n)}) \geq \frac{x^\gamma}{\gamma} exp\left(-\frac{\gamma}{1-\gamma}W^{(\theta_n)}(0,y) + \frac{1}{1-\gamma}W^{(\theta_{n+1})}(0,y)\right).$$

(**continue**)

Value Function-1

- Then

$$0 \leq W^{(\theta_n)}(0,y) - W(0,y) \leq \gamma^n (W^{(\theta_0)}(0,y) - W(0,y)).$$

- The value of θ_n-market

$$V^{(\theta_n)}(0,y) = \frac{x^\gamma}{\gamma} \exp(W^{(\theta_n)}(0,y)).$$

- The value of the original market

$$V(0,y) = \frac{x^\gamma}{\gamma} \exp(W(0,y)).$$

Proof of Theorem 4-3

- $J(T, x, y, \pi^{(n)}, c^{(n)})$ is given by

$$\frac{1}{\gamma} E\left[\int_0^T e^{-\rho t}(c^{(n)}(t, Y(t))X^{(n)}(t))^{\gamma} dt + e^{-\rho T}(X^{(n)}(T))^{\gamma}\right].$$

- We compare with $\phi^{\theta_{n+1}}(0, y) = \exp(\frac{1}{1-\gamma} W^{\theta_{n+1}}(0, y))$,

$$\phi^{\theta_{n+1}}(0, y) = E\left[\int_0^T e^{-\frac{\rho}{1-\gamma} t}(H^{\theta_{n+1}}(t))^{-\frac{\gamma}{1-\gamma}} dt + e^{-\frac{\rho}{1-\gamma} T}(H^{\theta_{n+1}}(T))^{-\frac{\gamma}{1-\gamma}}\right].$$

Proof of Theorem 4-2

- $X^{(n)}(t)$ is given by

$$\begin{aligned} X^{\theta_n}(t) = \quad & x \exp\left(\frac{1}{1-\gamma}\int_0^t \left\{(I_d - \sigma_f)\sigma_F^T D_y W^{\theta_n} + (\bar{\mu} - r\mathbf{1}_d)\right\}^T dB(s)\right) \\ & \cdot \exp\left(-\frac{1}{2}\frac{1}{(1-\gamma)^2}\int_0^t |(I_d - \sigma_f)\sigma_F^T D_y W^{\theta_n} + (\bar{\mu} - r\mathbf{1}_d))|^2 ds \right. \\ & \qquad \left. + \int_0^t \left(\frac{1}{1-\gamma}(\bar{\mu} - r\mathbf{1}_d)^T \sigma_F^T D_y W^{\theta_n} + \frac{1}{1-\gamma}|\bar{\mu} - r\mathbf{1}_d|^2\right) ds\right) \\ & \cdot \exp\left(\int_0^t r ds - \int_0^t e^{-\frac{1}{1-\gamma}W^{\theta_n}} ds\right) \end{aligned}$$

- $c^{(n)}(t) = \exp\left(-\frac{W^{\theta_n}(t, Y(t))}{1-\gamma}\right)$
- Use the equation and Ito's formula

$$\begin{aligned} & \partial_t W^{\theta_n} + \frac{1}{2}tr(a_F D_y^2 W^{\theta_n}) + \frac{1}{2(1-\gamma)} a_F^{ij} D_{y_i} W^{\theta_n} D_{y_j} W^{\theta_n} \\ & + (b + \frac{\gamma}{1-\gamma}\sigma_F \bar{\theta}_n)^T D_y W^{\theta_n} + \gamma r + \frac{\gamma}{2(1-\gamma)}|\bar{\theta}_n|^2 - \rho \\ & \qquad + (1-\gamma) e^{-\frac{1}{1-\gamma}W^{\theta_n}} = 0. \end{aligned}$$

Proof of Theorem 4-1

- We compare with

$$
\begin{aligned}
&(H^{\theta_{n+1}}(t))^{-\frac{\gamma}{1-\gamma}} \\
&= \exp\Big(\tfrac{\gamma}{1-\gamma}\int_0^t \{\theta_0 - \sigma_f\sigma_F^T D_y W^{\theta_n})\}^T dB(s) \\
&\qquad + \tfrac{\gamma}{2(1-\gamma)}\int_0^t \{|\theta_0|^2 + |\sigma_f\sigma_F^T D_y W^{\theta_n}|^2\} ds\Big)\cdot \exp\Big(\tfrac{\gamma}{1-\gamma}\int_0^t r\, ds\Big).
\end{aligned}
$$

  We omit $(s, Y(s))$ in the relation.

- By computation,

$$
\begin{aligned}
&\tfrac{1}{\gamma}(X^{\theta_n}(t)c^{\theta_n}(t, Y(t)))^\gamma \exp(-\rho t) \\
&= \tfrac{1}{\gamma}x^\gamma \exp\big(\tfrac{-\gamma}{1-\gamma}W^{\theta_n}(0,y)\big)(H^{\theta_{n+1}}(t))^{-\frac{\gamma}{1-\gamma}} \exp\big(-\tfrac{\rho}{1-\gamma}t + \int_0^t R(s)ds\big).
\end{aligned}
$$

- $R(s)$ is given by

$$
R(s) = \frac{\gamma^2}{2(1-\gamma)^2}|\overline{\theta}_n - \overline{\theta}_{n+1}|^2 > 0.
$$

Concluding Remarks-2

- The dynamic programming and martingale method are applied together to solve the problem.
- The dynamic programming is used to derive HJB equation.
- The HJB equation is rewritten as inf-sup type Isaacs equation.
- This is used to derive updating scheme.

Concluding Remarks-1

- Different from usual approach, where PDE theory provides analysis to prove the convergence.
- Using martingale method, we have expression for the value functions.
- This a key in our proof for the convergence.
- This works for the cases $0 < \gamma < 1$.
- For $\gamma < 0$, the argument in Theorem 4 does not work.
- We may need a different argument.