Data based Quantitative Analysis and Calculation under Nonlinear Expectations

Peng Shige

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- Nonlinearity is fundamental and necessary
- Worst case philosophy:

$$\hat{\mathbb{E}}[X] := \max_{\theta\in\Theta} E_{P_\theta}[X] :$$

$\hat{\mathbb{E}}$ becomes an sublinear operation, called sublinear expectation

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From linear to Nonlinear Expectations

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  - $\mathbb{E}[X_i] \downarrow 0$, if $X_i(\omega) \downarrow 0$, $\forall \omega$

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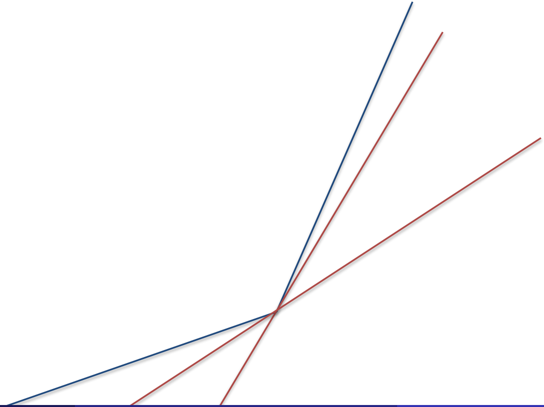

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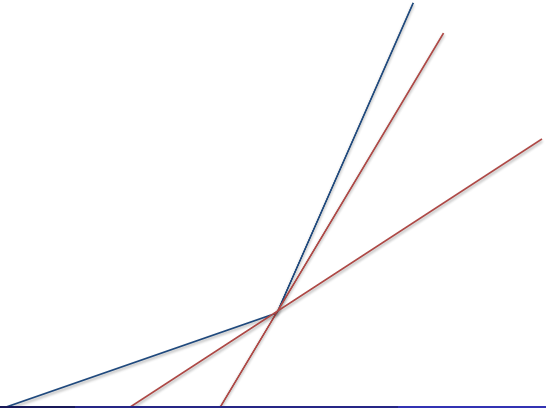

Expectation nonlinearity v.s. Probability & distribution uncertainty

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$$X \overset{d}{=} Y \quad \Longleftrightarrow \mathbb{E}[\varphi(X)] = \mathbb{E}[\varphi(Y)],$$

$$X \overset{d}{\geq} Y \quad \Longleftrightarrow \mathbb{E}[\varphi(X)] \geq \mathbb{E}[\varphi(Y)],$$

- Y is indenpendent of X if

$$\mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\mathbb{E}[\varphi(x, Y)]_{x=X}].$$

A toy but typical model: nonlinear Bernoulli model

- Randomly choose a ball from an urn (black box) containing Black and White balls ($W_1 + B_1 = 100$)
- But we know only $W_1 \in [40, 60]$
- $$\xi_1(\omega) = 1_{\{W_1 = \text{true}\}} - 1_{\{B_1 = \text{true}\}}$$

Repeat the above game: $i = 1, 2, 3, \cdots,$

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- Each time i, W_i is changed within $W_i \in [40, 60]$
- We get a sequence of random variables $\{\xi_i(\omega)\}_{i=1}^{\infty}$;
- This is a typical case of our daily uncertainty: envoronment changes all the time
- The output $\{\xi_i\}_{i=1}^{\infty}$ is i.i.d.:
  - $\xi_1, \xi_2, \cdots$ are identically distributed (same distribution uncertainty)
  - ξ_{i+1} is independent of $\{\xi_i\}_{i=1}^{n}$.
- $S_n = \sum_{i=1}^{n} \xi_i$: a nonlinear Bernoulli random walk.

A general random generator of nonlinear i.i.d. sequence $\{\xi_i\}_{i=1}^{\infty}$

- The 'black box' becomes a machine to output random vectors

$$\xi_i(\omega) \stackrel{d}{=} F_\theta, \quad i = 1, 2, 3, \cdots$$

but we only know:

$$\mathcal{L}(\xi_i) \in \{F_\theta\}_{\theta \in \Theta}, \quad i = 1, 2, 3, \cdots$$

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- $\xi_{i+1} \overset{d}{=} \xi_i, \quad \xi_{i+1}$ is independent of $(\xi_1, \xi_2, \cdots, \xi_i)$
- Examp.: $\{F_\theta\}_{\theta \in \Theta} = M_{[\underline{\mu}, \overline{\mu}]}$: all prob. distributions concentrated on $[\underline{\mu}, \overline{\mu}]$.

Important case: i.i.d. with a maximal distribution

- $M_{[\underline{\mu},\overline{\mu}]}$-distributed i.i.d sequence $\{\xi_i\}_{i=1}^{\infty}$: covers all possible distributions of sequences satisfying

$$\underline{\mu} \leq \xi_i(\omega) \leq \overline{\mu}, \quad i = 1, 2, 3, \cdots$$

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- Important task: how to narrow down $\overline{\mu} - \underline{\mu}$?
- Challenging objective: to establish a systematic framework $(\Omega, \mathcal{H}, \mathbb{E})$ compatible with $(\Omega, \mathcal{F}, P)$
- Important: hidden behind are PDEs (linear/nonlinear heat equation)!

Two fundamentally important nonlinear distributions

$$\xi_1 + \xi_2 \stackrel{d}{=} 2\xi_1 \quad \Longleftrightarrow \quad \xi_1 \stackrel{d}{=} M_{[\underline{\mu},\overline{\mu}]}$$

$$\xi_1 + \xi_2 \stackrel{d}{=} \sqrt{2}\xi_1 \quad \Longleftrightarrow \quad \xi_1 \stackrel{d}{=} N(0,[\underline{\sigma}^2,\overline{\sigma}^2])$$

Nonlinear Law of large number

Theorem (Peng2008-2010)

Let $\{Y_i\}_{i=1}^{\infty}$ *be i.i.d. sequence. Assume* $\mathbb{E}[|Y_1|^{1+\delta}] < \infty$.

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$$\lim_{n\to\infty} \mathbb{E}[\varphi(\frac{Y_1 + \cdots + Y_n}{n})] = \mathbb{E}[(\varphi(Y)] = \max_{v\in[\underline{\mu},\overline{\mu}]} \varphi(v).$$

where $\overline{\mu} = \mathbb{E}[Y_1]$, $\underline{\mu} = -\mathbb{E}[-Y_1]$.

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$$Y \overset{d}{=} M_{[\overline{\mu},\underline{\mu}]}: \text{ Maximal distribution.}$$

$u(x,t) := \mathbb{E}[\varphi(x + (1-t)Y)]$ solves the 1st order PDE:

$$\partial_t u(t,x) + g(\partial_x u) = 0, \quad u(1,x) = \varphi(x)$$

Nonlinear Central limit theorem

Theorem (Peng2008-2010)

Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. sequence. We assume furthermore that

$$\mathbb{E}[|X_1|^{2+\delta}] < \infty \quad \mathbb{E}[X_1] = \mathbb{E}[-X_1] = 0$$

Then, for each $\varphi \in C_b(\mathbb{R})$,

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where X is $N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$-distributed.

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$u(x,t) := \mathbb{E}[\varphi(x + \sqrt{1-t}X)]$ solves the 2nd order PDE

Nonlinear normal distributions

Definition

A random variable X in $(\Omega, \mathcal{H}, \mathbb{E})$ is normal if the function

$$u(t,x) := \mathbb{E}[\varphi(x + \sqrt{1-t}X)]$$

is the solution of the nonlinear PDE

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where

- $G(a) = \frac{1}{2}[\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-]$ $\quad \overline{\sigma}^2 = \mathbb{E}[X^2], \quad \underline{\sigma}^2 = -\mathbb{E}[-X^2]$

A General misunderstanding caused from

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Classical 'Monté-Carlo' approach for estimating $\hat{\mathbb{E}}[\varphi(X)]$ through data

- Key point: How to obtain $\hat{\mathbb{E}}[\varphi(X)]$ through its sample $\{x_i\}_{i=1}^N$?
- In many practice cases: we care about $\hat{\mathbb{E}}[\varphi(X)]$ with a specific function $\varphi(x)$:
  a consumption utility function, a contract, a cost function ....
- In a classical probability space $(\Omega, \mathcal{F}, P)$, we can apply LLN to calculate

$$E[\varphi(X)] \sim \mathbb{M}[\varphi(X)] := \frac{1}{N}\sum_{i=1}^{N} \varphi(x_i)$$

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- But: Is $\{x_i\}_{i=1}^N$ a classical i.i.d.?

Papers of limit theory with nonlinear expectations

- Convergence to a self-normalized G-Brownian motion, Zhengyan Lin and Zhixin Zhang, in PUQR, 2017.
- Nutz &van Handel (2013) Constructing sublinear expectations on path space. SPA, 123(8)
- P. 2006: G-expectation, G-Brownian motion and related stochastic calculus of Ito's type. The Abel Symposium 2005, Springer.
- P. (2008) A new central limit theorem under sublinear expectations
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Papers of limit theory with nonlinear expectations

- P. (2010) Nonlinear Expectations and Stochastic Calculus under Uncertainty (2010a). Preprint: arXiv:1002.4546 [math.PR]
- P. (2010) Tightness, weak compactness of nonlinear expectation (arXiv),
- Yan & Soner (2012) Weak approximation of G-expectations SPA,
- Zhang (2015) Donsker's invariance principle under the sublinear expectation with an application to Chung's law of the iterated logarithm. Commun. Math. Stat .
- Zhang,: (2016) Rosenthal's inequalities for independent and negatively dependent random variables under sub-linear expectations with applications. Sci. China Math.
- Fang X., Peng, S. & Shao Q. working paper

φ-max-mean algorithm: the data-based distribution of X

- Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ be a sublinear expectation space and

$$\{x_i\}_{i=1}^{n\times m}: \quad \text{i.i.d. sample of a random vector} \quad X$$

- The max-mean algorithm to estimate $\hat{\mathbb{E}}[\varphi(X)]$:

$$\hat{\mathbb{M}}[\varphi] = \max\{Y_n^k : k = 1, \cdots, m\},$$

where

$$Y_n^k = \frac{1}{n}\sum_{i=1}^{n} \varphi(x_{n(k-1)+i}).$$

- By nonlinear LLN, when $n \to \infty$, $\{Y_n^k\}_{k=1}^m \overset{d}{\Longrightarrow}$ an i.i.d. $\{Y^k\}_{k=1}^m$,

$$Y^k \overset{d}{=} M([\underline{\mu}_{\varphi(X)}, \overline{\mu}_{\varphi(X)}]).$$

- But $\max\{Y^k : k = 1, \cdots, m\}$ provides us the assymptotically optimal unbiased estimate of $\overline{\mu}_{\varphi(X)} = \hat{\mathbb{E}}[\varphi(X)]$.

Optimality of the estimate

The optimality of the above estimate is based on the following quite simple, but very fundamental result:

Theorem (Jin-Peng2016)

Let $Y^1, \cdots, Y^m$ be i.i.d. and maximally distributed:

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$$\underline{\mu} \le \min\{Y^1(\omega), \cdots, Y^n(\omega)\} \le \max\{Y^1(\omega), \cdots, Y^n(\omega)\} \le \overline{\mu}.$$

Moreover

$$\widehat{\overline{\mu}}_n = \max\{Y^1, \cdots, Y^n\},$$

is the maximum unbiased estimate of $\overline{\mu}$.

Parameter estimates for nonlinear distributed model

- Many typical nonlinear distributions

$$M_{[\underline{\mu},\overline{\mu}]}, \quad N(\mu, [\underline{\sigma}^2, \overline{\sigma}^2]), \quad P_{[\underline{\lambda},\overline{\lambda}]} \text{ (Nonlinear Poisson)}$$

- asymptotically unbiased estimates: ,

$$\hat{\underline{\sigma}}^2 := \min_{1\le k\le m} \sigma_k^2, \qquad \hat{\overline{\sigma}}^2 := \max_{1\le k\le m} \sigma_k^2$$

$$\text{where} \quad \sigma_k^2 := \frac{1}{n}\sum_{j=1}^{n}(x_{n(k-1)+j} - \mu)^2.$$

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Theorem.

If $(B_t)_{t\geq 0}$ is a G–**Brownian motion** and $\mathbb{E}[B_t] = \mathbb{E}[-B_t] \equiv 0$ then:
$B_{t+s} - B_s \stackrel{d}{=} N(0, [\underline{\sigma}^2 t, \overline{\sigma}^2 t])$, $\forall\ s, t \geq 0$ □

Real case study: from VaR to GVaR Problem challenged by CFFEX (China Financial Future Exchange)

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$$
\begin{aligned}
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Real case study: from VaR to GVaR Problem challenged by CFFEX (China Financial Future Exchange)

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F_G has the explicit expression:

$$F_G(x) = \int_{-\infty}^{x} \frac{\sqrt{2}}{\sqrt{\pi(\overline{\sigma} + \underline{\sigma})^2}} \left[\exp(\frac{-y^2}{2\overline{\sigma}^2})\mathbf{1}_{y\leq 0} + \exp(\frac{-y^2}{2\underline{\sigma}^2})\mathbf{1}_{y>0}\right] dy. \quad (3)$$

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- For each $\bar{t}$, use the passed 1 year data $\{X_{\bar{t}-s}\}_{0 \le s \le l-1}$ to estimate two parameters $\underline{\sigma}_{\bar{t}}^2$ and $\overline{\sigma}_{\bar{t}}^2$ at the day $\bar{t}$:

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- Fix a window width $w = 100$ use the moving window

$$\sigma_{\bar{t},w}^2 := \sigma^2(X_{\bar{t}-w+1}, \cdots, X_{\bar{t}}).$$

- Then get the upper and low data variances:

$$\overline{\sigma}_{\bar{t}}^2 = \max\{\sigma_{\bar{t},20}^2, \sigma_{\bar{t}-s,w}^2, ;\ s \in [0, \cdots, l-w]\},$$
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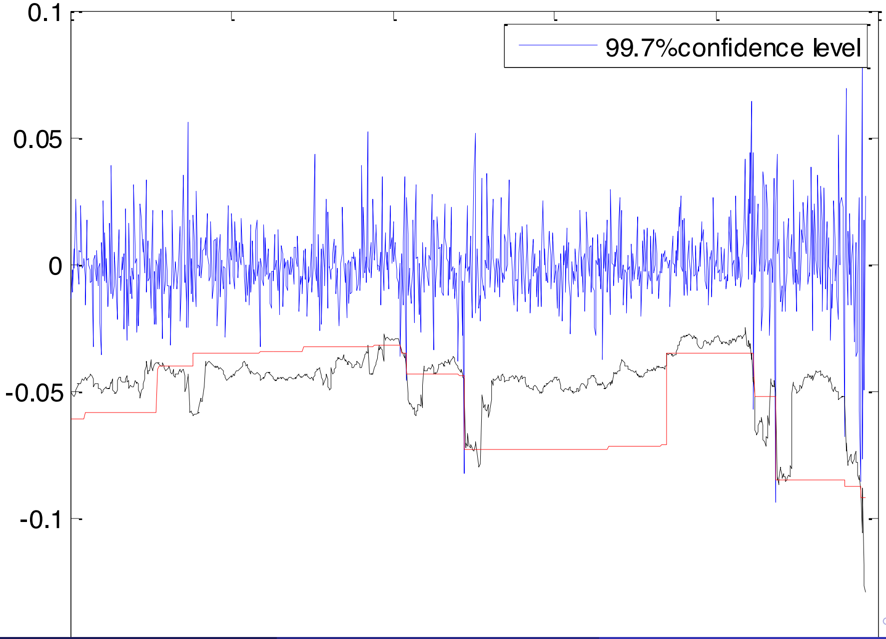
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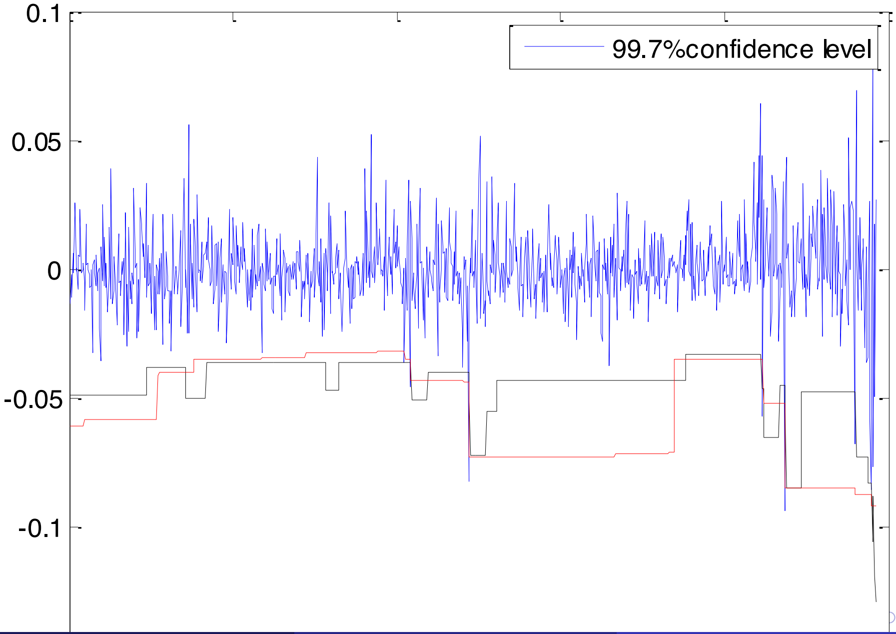
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Theoretical prospective

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- Nonlinear expectation theory: non-Euclidean's Axiomatization
- for uncertainty quantifications under probability-distribution uncertainties.

Probability Space	Nonlinear Expectation Space
$(\Omega, \mathcal{F}, P)$	$(\Omega, \mathcal{H}, \mathbb{E})$: (sublinear is basic)
Distributions: $X \overset{d}{=} Y$	
Independence: Y indep. of X	
LLN and CLT	
Normal distributions	
Brownian motion $B_t(\omega) = \omega_t$	
Quadratic variable. $\langle B \rangle_t = t$	
Lévy process	

Probability v.s. Nonlinear Expectation

Probability Space	Nonlinear Expectation Space
$(\Omega, \mathcal{F}, P)$	$(\Omega, \mathcal{H}, \mathbb{E})$: (sublinear is basic)
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Probability v.s. Nonlinear Expectation

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Distributions: $X \overset{d}{=} Y$	$X \overset{d}{=} Y$,
Independence: Y indep. of X	Y indep. of X, (non-symm.)
LLN and CLT	
Normal distributions	
Brownian motion $B_t(\omega) = \omega_t$	
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Probability v.s. Nonlinear Expectation

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Independence: Y indep. of X	Y indep. of X, (non-symm.)
LLN and CLT	nonlinear LLN and CTL
Normal distributions	
Brownian motion $B_t(\omega) = \omega_t$	
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Normal distributions	nonlinear Normal distributions
Brownian motion $B_t(\omega) = \omega_t$	Nonlinear B.M. $B_t(\omega) = \omega_t$,
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Lévy process	

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Lévy process	nonlinear Lévy process

Probability v.s. Nonlinear Expectation

Probability Space	Nonlinear Expectation Space
Itô's calculus for BM	Itô's calculus for nonlinear BM
SDE $dx_t = b(x_t)dt + \sigma(x_t)dB_t$	
Diffusion: $\partial_t u - \mathcal{L}u = 0$	
Markovian pro. and semi-grou	
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$E[X\|\mathcal{F}_t] = E[X] + \int_0^T z_s dB_s$	

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Diffusion: $\partial_t u - \mathcal{L}u = 0$	$\partial_t u - G(t, x, u, Du, D^2u) = 0$
Markovian pro. and semi-grou	Nonlinear Markovian
Martingales	Nonlinear Martingales
$E[X\|\mathcal{F}_t] = E[X] + \int_0^T z_s dB_s$	$\mathbb{E}[X\|\mathcal{F}_t] = \mathbb{E}[X] + \int_0^t z_s dB_s + K_t$

Probability v.s. Nonlinear Expectation

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Martingales	Nonlinear Martingales
$E[X\|\mathcal{F}_t] = E[X] + \int_0^T z_s dB_s$	$\mathbb{E}[X\|\mathcal{F}_t] = \mathbb{E}[X] + \int_0^t z_s dB_s + K_t$ $K_t = \int_0^t \eta_s d\langle B\rangle_s - \int_0^t 2G(\eta_s)ds$

New journal : Probability, Uncertainty and Quantitative Risk

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Thank you 谢谢