Solvability of a Class of Dirichlet Problems

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Outline

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Solvability question

- Eq - Consider an equation, for $\alpha \in (0, 2)$

$$|\nabla u| + (-\Delta)^{\alpha/2} u + u - 1 = 0, \text{ on } B_1, \quad u = 0, \text{ on } B_1^c.$$

- If $\alpha \in [1, 2]$, it is solvable, see
  [BCI08] G. Barles, et. al. On the Dirichlet problem for second-order elliptic integro-differential equations. Indiana Univ. Math.

- Q - For $\alpha \in (0, 1)$, is this solvable?

Outline

HJB

- Eq - Note that $|p| = \sup_{a\in B_1} -a\cdot p$, hence rewrite

$$|\nabla u| + (-\Delta)^{\alpha/2}u + u - 1 = 0, \text{ on } B_1, \quad u = 0, \text{ on } B_1^c.$$

into HJB

$$\sup_{a\in B_1} -\mathcal{L}^a u + u - 1 = 0, \text{ on } B_1, \quad u = 0, \text{ on } B_1^c.$$

where

$$\mathcal{L}^a = a\cdot\nabla u - (-\Delta)^{\alpha/2}u.$$

- HJB is associated to Exit control problem ...

Exit control problem

- L is a symmetric α-stable process,
- A dynamic controlled by $m \in C^{0,1}(\mathbb{R}^d, B_1)$ is

$$X_t = x + \int_0^t m(X_t)dt + L_t;$$

- First exit time: $\tau = \inf\{t > 0, X_t \notin B_1\} := T_{B_1^c}(X)$.
- The value function V is defined as

$$V(x) = \inf_m \mathbb{E}^{m,x}\Big[\int_0^\tau e^{-s} \cdot 1 \, ds\Big] := \inf_m V_m(x).$$

Dynamic programing approach

A possible approach to the solvability

- V satisfies dynamic programing principle (DPP)

$$V(x) = \inf_m \mathbb{E}^{m,x}\Big[\int_0^h e^{-s} \cdot 1\, ds + e^{-h} V(X_h)\Big], \forall h \in (0, \tau).$$

- Formally, it leads to

$$\sup_{a \in B_1} -\mathcal{L}^a V + V - 1 = 0, \text{ on } B_1, \quad u = 0, \text{ on } B_1^c.$$

provided that V satisfies continuity and DPP.

- However, ...

DPP may not help

Toy model

- Consider exit control problem

$$V(x) = \inf_{m \in C^{0,1}(\mathbb{R}^d; B_1)} \mathbb{E}^{m,x}\Big[\int_0^\tau e^{-s} \cdot (-1) ds\Big]$$

with $\tau = T_{B_1^c}(X)$ of controlled dynamic

$$X_t = x + \int_0^t m(X_s) ds.$$

- Formally DPP says, HJB is solvable, i.e. V solves

$$|\nabla u| + u + 1 = 0, \text{ on } B_1, \quad u = 0, \text{ on } \partial B_1.$$

- But, if $u \in C^1(B_1) \cap C(\bar{B}_1)$, it gives contradiction by

$$u = -|\nabla u| - 1 \leq -1, \text{ on } B_1.$$

- One can show there is no viscosity solution, and DPP shall not help.

Outline

Alternative approach

Overview of the proof on the solvability

We attempt the followings for the solvability question:

1. (CP) If u and v are sub and supersolution, then $u \leq v$.
2. (PM) If there exist subsolution and supersolution, then there exists a unique solution.

Example 4.6. Theorem 4.1 leaves open the question of when a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of (DP) that vanish on $\partial\Omega$ can be found. Let us consider this problem for the equation

Figure: Quoted from P25 of *User's guide to viscosity solution*

Subsolution

- $u = 0$ is a subsolution, since

$$\text{if } u = 0, \quad \text{then } |\nabla u| + (-\Delta)^{\alpha/2} u + u - 1 = -1 \leq 0.$$

Hunting for supersolution

- 1 - The first clue

Recall that, V_m is defined as

$$V_m(x) = \mathbb{E}^{m,x}\Big[\int_0^\tau e^{-s} \cdot 1\, ds\Big] = \mathbb{E}^{m,x}[1 - e^{-\tau}].$$

- Fact - If $V_m \in C(\bar{O})$ with $V_m = 0$ on ∂O, then V_m is a supersolution.

- Pf - Strong Markov property and Ito's formula.

- Clue - Find one control m so that $V_m \in C(\bar{O})$ ...

Hunting for supersolution

- 2 - Exit time

Let $y \to x$ and we try to show $V_m(y) \to V_m(x)$.

- Note that the mappings are, with Skorohod space $\mathbb{D}$

$$(\Omega, \mathbb{P}) \xrightarrow{X} (\mathbb{D}, \mathbb{P}^{m,x} X^{-1}) \xrightarrow{\tau} (\mathbb{R}, \mathbb{P}^{m,x} X^{-1} \tau^{-1})$$

- X is Feller, and hence $\mathbb{P}^{m,y} X^{-1} \Rightarrow \mathbb{P}^{m,x} X^{-1}$
- If $\tau : \mathbb{D} \mapsto \mathbb{R}$ is continuous, then

$$\mathbb{P}^{m,y} X^{-1} \tau^{-1} \Rightarrow \mathbb{P}^{m,x} X^{-1} \tau^{-1}.$$

- $V_m(x) = \mathbb{E}^{m,x}[1 - e^{-\tau}] := \mathbb{E}^{m,x} f(\tau)$ with $f \in C_b$, we have

$$V_m(y) = \mathbb{E}^{m,y} f(\tau) \to \mathbb{E}^{m,x} f(\tau) = V_m(x).$$

Hunting for supersolution

- 3 - Is exit time continuous? It's just a wish.

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although g, k and f are smooth functions, u need not even be continuous, due to the discontinuous nature of the functional $\tau(\omega)$.

Actually, instead of considering τ one can consider τ', the first exit time of the path $x(\cdot, \omega)$ from $\bar{G}$. There is no *a priori* reason why one should prefer one to the other. We have a similar formula for the solution

(2.10)

$$v(x) = E^{P_x}\left[\int_0^{\tau'} \exp\left\{-\int_0^t k(x(s))\,ds\right\} f(x(t))\,dt + \exp\left\{-\int_0^{\tau'} k(x(s))\,ds\right\} g(x(\tau'))\right]$$

Figure: A remark from [Sroock and Varadhan 1972]

Hunting for supersolution

- 4 - Not just a wish

Let $y \to x$ and we try to show $V_m(y) \to V_m(x)$.

- If $\tau : \mathbb{D} \mapsto \mathbb{R}$ was continuous, then we are done. Alternatively,
- If there exists $\Gamma \subset \mathbb{D}$ such that
  - $\tau : \mathbb{D} \mapsto \mathbb{R}$ is continuous on Γ;
  - $\mathbb{P}^{m,x} X^{-1}(\Gamma) = 1$ or $\mathbb{P}^{m,x}(X \in \Gamma) = 1$;

  then continuous mapping theorem says

$$\mathbb{P}^{m,y} X^{-1} \tau^{-1} \Rightarrow \mathbb{P}^{m,x} X^{-1} \tau^{-1}.$$

- So, we have $V_m(y) = \mathbb{E}^{m,y} f(\tau) \to \mathbb{E}^{m,x} f(\tau) = V_m(x)$ with $f \in C_b$.

Hunting for supersolution

- 5 -Hunting for Γ

- Lem - The mapping $\tau : \mathbb{D} \mapsto \mathbb{R}$ is continuous on $\Gamma = \Gamma_1 \cap \Gamma_2$, where

$$\Gamma_1 = \{\omega : \omega \text{ is continuous at } \tau(\omega^-)\}, \quad \Gamma_2 = \{\omega : \bar{\tau} = \tau\}.$$

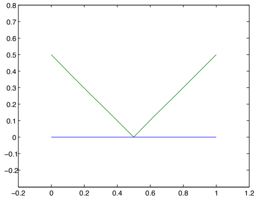

Figure: $\omega \notin \Gamma_2$

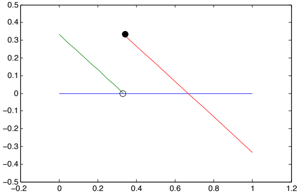

Figure: $\omega \notin \Gamma_1$

Hunting for supersolution

- 6 - The boundary

- τ is continuous on Γ;
- $\mathbb{P}^{m,x}(\Gamma) = 1$, b/c
  - Meyer's Theorem says $\mathbb{P}^{m,x}(\Gamma_1) = 1$;
  - If O is regular w.r.t. $\mathbb{P}^{m,x}$, then it implies $\mathbb{P}^{m,x}(\Gamma_2) = 1$.

- Next - We search for m such that

$$\mathbb{P}^{m,x}(\bar{\tau} = 0) = 1, \ \forall x \in \partial O$$

where

$$\bar{\tau} = \inf\{t > 0, X_t \notin \bar{O}\}.$$

Plainly, we say ∂O is regular w.r.t. X^m

Hunting for supersolution

- 7 - Regular boundary

- Ex - Let $O = (-\infty, 0)$, then $\partial O = \{0\}$ is regular w.r.t. W.
(i.e. If you run BM from 0, it hits positive half line immediately.)

- Pf -

$$\mathbb{P}(\bar{\tau} = 0) = \lim_{t \to 0} \mathbb{P}(\bar{\tau} \leq t) \geq \lim_{t \to 0} \mathbb{P}(W_t > 0) \geq 1/2.$$

B/c $\{\bar{\tau} = 0\} \in \mathcal{F}_0^+$, the conclusion follows from Bluementhal 0-1 Law.

- Fact - ∂B_1 is regular w.r.t. symmetric α-stable process L^α for any $\alpha \in (0, 2)$.

Hunting for supersolution

- 8 - When Messi touches the ball, what is the $\mathbb{P}\{\text{Messi kicks the ball in}\}$?

- Fact - Let $\mathcal{F}_s = \sigma(W_r : r \leq s)$. If $A \in \mathcal{F}_0^+$, then $\mathbb{P}(A) \in \{0, 1\}$.

Hunting for supersolution

- 8 - Mission is completed

- Eq - With $\alpha \in (0, 2]$

$$|\nabla u| + (-\Delta)^{\alpha/2} u + u - 1 = 0, \text{ on } B_1, \quad u = 0, \text{ on } B_1^c.$$

- Associated process is

$$X_t = x + \int_0^t m(X_s) e_1 ds + L_t$$

- $m(x) = 0 \in \mathcal{M}$ makes ∂B_1 is regular w.r.t. X_t.

- A - Supersolution exists, so does the solution. Hooray!

Outline

Conclusion

The work can be extended. (arXiv)

Consider

$$F(u,x) + u(x) - \ell(x) = 0,\ x \in O,\ \text{with } u(x) = 0,\ x \in \bar{O}^c.$$

In the above, the operator is defined via

1. $F(u,x) = -\inf_{a\in\Lambda} H(u,x,a) - \mathcal{I}(u,x)$
2. $H(u,x,a) = \frac{1}{2}tr(A(a)D^2u) + b(a)\cdot Du$ with $A(a) = \sigma'(a)\sigma(a)$.
3. $\mathcal{I}(u,x) = \int_{B_1}(u(x+y) - u(x) - Du(x)\cdot yI_{B_1}(y))\nu(dy)$;

Theorem. Under standing assumptions, there exists a solution, if

$$\text{(A1) } \mathcal{M} \neq \emptyset; \quad \text{and} \quad \text{(A2) } \ell \geq 0.$$

where $\mathcal{M} = \{m \in C^{0,1}(\mathbb{R}^d, \Lambda) : \mathbb{P}^{m,x}(\bar{\tau} = 0) = 1,\ \forall x \in \partial O\}$.