Semi-Static Hedging and Pricing American Floating Strike Lookback Options

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April 25, 2017

Contribution

Contributions on Pricing

- We extend Derman et al. (1995) and Carr et al. (1998) to price American floating strike lookback options based a hypothetic static hedging (HSHP) by a change of numeraire.

- HSHP for pricing is generally superior to the tree method of Babbs (2000) and it has the pattern of monotonic convergence.

Contributions on Replication

- Apply put-call symmetry to transform HSHP into a semi-static hedging portfolio (SSHP) so that replication of the American floating strike lookback options is feasible in reality.

- The numerical analyses indicate that the hedging performance of SSHP is far more stable than that of a delta-hedged portfolio with daily rebalance.

Introduction

- Why exotic options are so important?
  - They can serve as suitable tools for risk management.
  - Investors can use barrier options or floating strike lookback options to protect themselves from extreme price.

- Literature in pricing floating strike lookback options.
  - Only tree methods such as Babbs (2000) can be applied to price American floating strike lookback options.

- Why it is hard to replicate an American floating strike lookback option in a static manner?
  - Because its payoff of depends both on the realized maximum or minimum of the underlying asset and the stock price.

- Why semi-static hedge for American floating strike lookback option is feasible?
  - European call and put prices are homogeneous of degree one in stock price and strike price with the put-call symmetry holds for European options.

Advantages of purposed method

- The recalculation of the option prices under the HSHP is much easier and quicker than the tree methods.
- The SSHP needs to be rebalanced only when the realized maximum of the stock price changes.
- The calls and puts of the rebalanced portfolio have different strike prices depending on the new realized maximum of stock price but with the same weights.

Pricing by HSHP

- The payoff of an American floating strike lookback put

$$max(S_{max}(T) - S(T))$$

where $S_{max}(t) = max(S_a; t_0 \leq a \leq t)$, t_0 is the initial time.

- The PDE of the American floating strike lookback put

$$\frac{\sigma^2 S^2}{2}\frac{\partial^2 P^A_{Lookback}}{\partial S^2} + (r-q)S\frac{\partial P^A_{Lookback}}{\partial S} + \frac{\partial P^A_{Lookback}}{\partial t} = rP^A_{Lookback}, (1)$$

where $S^*(t) \leq S(t) \leq S_{max}(t)$ and $S^*(t)$ is the early exercise boundary.

- The price of an American floating strike lookback put option using the underlying asset as numeraire

$$V_{Lookback}^{A}(u(t),1,t)=\frac{P_{Lookback}^{A}(S_{max}(t),S(t),t)}{S(t)}, \qquad (2)$$

where $u(t)=S_{max}(t)/S(t)$.

- The PDE of $V_{Lookback}^{A}(u(t),1,t)$

$$\frac{\sigma^2 u^2}{2}\frac{\partial^2 V_{Lookback}^{A}}{\partial u^2}+(q-r)u\frac{\partial V_{Lookback}^{A}}{\partial u}+\frac{\partial V_{Lookback}^{A}}{\partial t}=qV_{Lookback}^{A}, \qquad (3)$$

where $1\le u(t)<u^*(t)$, $u^*(t)=\frac{S_{max}(t)}{S^*(t)}$. (a world where the risk -free rate equals to q and a continuous dividend yield equals to r).

Boundary conditions of the PDE of $V^{A}_{Lookback}(u(t),1,t)$

$$V^{A}_{Lookback}(u(T),1,T) = \max(u(T)-1,0), \tag{4}$$

$$V^{A}_{Lookback}(u^{*}(t),1,t) = u^{*}(t)-1, \tag{5}$$

$$\left.\frac{\partial V^{A}_{Lookback}(u(t),1,t)}{\partial u(t)}\right|_{u(t)=u^{*}(t)} = 1, \tag{6}$$

and

$$\left.\frac{\partial V^{A}_{Lookback}(u(t),1,t)}{\partial u(t)}\right|_{u(t)=1^{+}} = 0. \tag{7}$$

How to price $V^{A}_{Lookback}(u(t), 1, t)$?

- Vanilla European options in this world satisfy the same PDEs as (3), so $V^{A}_{Lookback}(u(t), 1, t)$ is similar to an American call with strike price equal to one in such a world.
- Use a set of hypothetic vanilla European options to replicate $V^{A}_{Lookback}(u(T), 1, T)$ by satisfying the same boundary conditions
- When $u(t)$ is equal to $u^{*}(t)$ at time t_{n-1}, the value-matching condition (5) and smooth-pasting condition (6) imply that

$$u^{*}_{n-1} - 1 = C^{E}(u^{*}_{n-1}, 1, \sigma, q, r, T - t_{n-1})$$

$$+w_{n-1}C^E(u_{n-1}^*, u_{n-1}^*, \sigma, q, r, T - t_{n-1})$$

$$+\widehat{w}_{n-1}P^E(u_{n-1}^*, 1, \sigma, q, r, T - t_{n-1}), \tag{8}$$

$$1 = \Delta_C^E(u_{n-1}^*, 1, \sigma, q, r, T - t_{n-1})$$

$$+w_{n-1}\Delta_C^E(u_{n-1}^*, u_{n-1}^*, \sigma, q, r, T - t_{n-1})$$

$$+\widehat{w}_{n-1}\Delta_P^E(u_{n-1}^*, 1, \sigma, q, r, T - t_{n-1}), \tag{9}$$

$$0 = \Delta_C^E(1,1, \sigma, q, r, T - t_{n-1})$$

$$+w_{n-1}\Delta_C^E(1, u_{n-1}^*, \sigma, q, r, T - t_{n-1})$$

$$+\widehat{w}_{n-1}\Delta_P^E(1,1,\sigma,q,r,T-t_{n-1}\ )\ ,$$

(10)

■ Follow the similar process and work backward to determine the set of functions which replicate the boundary conditions of $V_{Lookback}^{A}(u(t),1,t)$, which states as follows:

$$\begin{aligned}
&V_{Lookback}^{A}(u(t),1,t)\\
&= C^E(u(t),1,\sigma,q,r,T-t)\\
&\qquad +w_{n-1}C^E(u(t),u_{n-1}^{*},\sigma,q,r,T-t)\\
&\qquad +\widehat{w}_{n-1}P^E(u(t),1,\sigma,q,r,T-t)\\
&\qquad +w_{n-2}C^E(u(t),u_{n-2}^{*},\sigma,q,r,t_{n-1}-t)\\
&\qquad +\widehat{w}_{n-2}P^E(u(t),1,\sigma,q,r,t_{n-1}-t)\\
&\qquad +\cdots+w_0C^E(u(t),u_0^{*},\sigma,q,r,t_1-t)
\end{aligned}$$

$$+\widehat{w}_0 P^E(u(t), 1, \sigma, q, r, t_1 - t)$$

(15)

- By changing the numeraire back to cash, we can obtain the option value as follows:

$$P^A_{Lookback}(S_{max}(t), S(t), t) = V^A_{Lookback}(u(t), 1, t)S(t). \qquad (16)$$

Numerical Results of HSHP

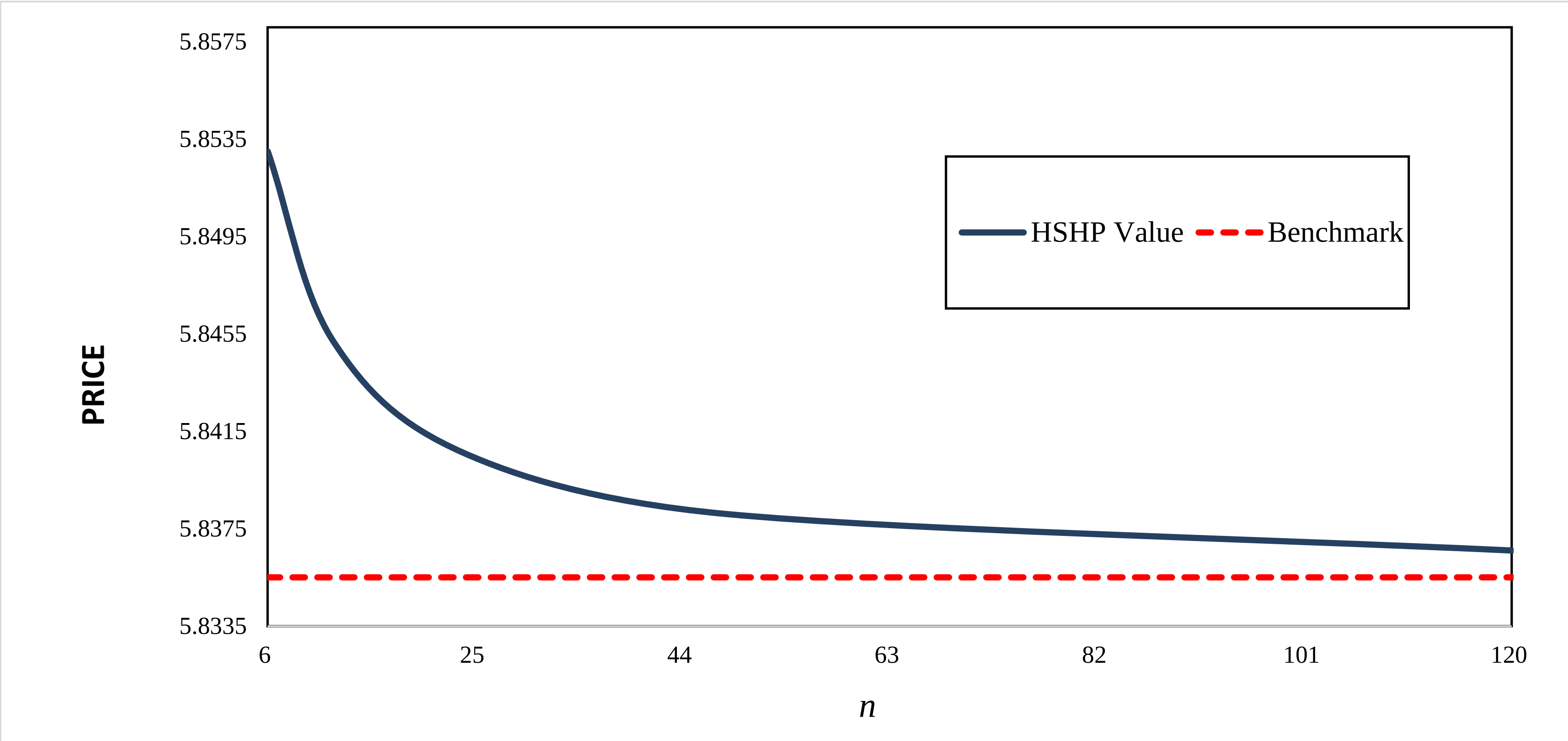
- The convergence of the HSHP values
  - We use the parameter setting of Chang et al. (2007) to choose: $S(t) = 50$, $u(t) = S_{max}(t)/S(t)$ =1.02, $r = 0.05$, $q = 0.05$, $T = 0.5$, and $\sigma = 0.2$.
  - The benchmark (5.8355) using the pricing formula of Conze (1991) and a control variate adjustment based on Babbs (2000):

Benchmark = Conze(1991) European Closed-Form Option Value

$$+ \text{Babbs American Option Value}$$

$$-\text{Babbs (2000) European Option Value} \tag{17}$$

Fig 1. The Convergence of the HSHP Values for Pricing



We use the parameter setting of Chang et al. (2007) to choose: $S(t) = 50$, $u(t) = S_{max}(t)/S(t)$ =1.02, $r = 0.05$, $q = 0.05$, $T = 0.5$, and $\sigma = 0.2$. The benchmark accurate American floating lookback put option price which uses Babbs's method with 1,000 times step per day is 5.8355.

Efficiency analysis in purposed method and tree method

- We price 36 American floating strike lookback put options.
- The root mean squared errors (RMSEs) of the HSHP are 0.0306, 0.0165, and 0.0087 for hedging frequency $n =$ 6, 12, and 24, respectively. The errors are reduced almost by a half when n, becomes doubled.
- The results are also similar in the root mean squared relative errors (RMSREs).

- However, the RMSEs and RMSREs of Babbs (2000) with Δ_t=1/960, 1/1920, and 1/3840 show the oscillatory pattern.
- When the time steps per day is doubled, the computation time becomes almost four times in tree method. However, when *n* is doubled, the computation time becomes almost double in HSHP.

Table 1. The valuation based on tree approach and the HSHP

σ	*T*	Tree			HSHP			Benchmark
		$\Delta_t = \frac{1}{960}$	$\Delta_t = \frac{1}{1920}$	$\Delta_t = \frac{1}{3840}$	$n = 6$	$n = 12$	$n = 24$	
Panel A. $r = 0.025; q = 0.05$								
0.1	0.1	1.5938	1.5963	1.5954	1.5983	1.5969	1.5962	1.5955
0.1	0.3	2.5562	2.5576	2.5570	2.5687	2.5631	2.5602	2.5571
0.1	0.5	3.2873	3.2884	3.2880	3.3091	3.2990	3.2937	3.2880
0.2	0.1	2.8403	2.7606	2.7595	2.7667	2.7637	2.7620	2.7603
0.2	0.3	4.7923	4.7428	4.7421	4.7647	4.7543	4.7487	4.7426
0.2	0.5	6.2072	6.1672	6.1667	6.2055	6.1875	6.1776	6.1670
0.4	0.1	5.3628	5.3585	5.4027	5.3791	5.3706	5.3660	5.3611
0.4	0.3	9.4836	9.4812	9.5086	9.5422	9.5145	9.4993	9.4827
0.4	0.5	12.4380	12.4363	12.4584	12.5415	12.4931	12.4665	12.4374
Panel B. $r = 0.05; q = 0.05$								
0.1	0.1	1.5125	1.5150	1.5140	1.5146	1.5145	1.5145	1.5141
0.1	0.3	2.3403	2.3418	2.3411	2.3431	2.3422	2.3418	2.3411
0.1	0.5	2.9387	2.9399	2.9393	2.9430	2.9414	2.9405	2.9393
0.2	0.1	2.7677	2.6909	2.6898	2.6934	2.6922	2.6915	2.6906

0.2	0.3	4.5882	4.5420	4.5413	4.5515	4.5471	4.5446	4.5417
0.2	0.5	5.8724	5.8358	5.8352	5.8530	5.8452	5.8406	5.8355
0.4	0.1	5.2963	5.2916	5.3347	5.3077	5.3014	5.2980	5.2939
0.4	0.3	9.2824	9.2797	9.3095	9.3262	9.3055	9.2938	9.2808
0.4	0.5	12.1018	12.0997	12.1207	12.1804	12.1439	12.1234	12.1004
σ	*T*	Tree			HSHP			Benchmark
		$\Delta_t = \frac{1}{960}$	$\Delta_t = \frac{1}{1920}$	$\Delta_t = \frac{1}{3840}$	$n = 6$	$n = 12$	$n = 24$	
Panel C. *r = 0.05; q = 0.025*								
0.1	0.1	1.4597	1.4617	1.4605	1.4581	1.4594	1.4599	1.4606
0.1	0.3	2.2108	2.2118	2.2110	2.2023	2.2067	2.2088	2.2110
0.1	0.5	2.7433	2.7440	2.7433	2.7289	2.7363	2.7398	2.7434
0.2	0.1	2.7184	2.6429	2.6415	2.6408	2.6417	2.6419	2.6420
0.2	0.3	4.4637	4.4190	4.4181	4.4149	4.4171	4.4179	4.4183
0.2	0.5	5.6841	5.6492	5.6485	5.6439	5.6470	5.6481	5.6485
0.4	0.1	5.2531	5.2478	5.2899	5.2585	5.2544	5.2521	5.2494
0.4	0.3	9.1735	9.1702	9.1954	9.2007	9.1877	9.1799	9.1707
0.4	0.5	11.9441	11.9415	11.9615	11.9958	11.9721	11.9581	11.9417
Panel D. *r = 0.05; q = 0*								
0.1	0.1	1.4154	1.4171	1.4159	1.4109	1.4131	1.4146	1.4160
0.1	0.3	2.1065	2.1073	2.1064	2.0890	2.0977	2.1021	2.1066
0.1	0.5	2.5906	2.5911	2.5904	2.5615	2.5758	2.5830	2.5906
0.2	0.1	2.6738	2.5995	2.5980	2.5931	2.5959	2.5972	2.5982
0.2	0.3	4.3545	4.3113	4.3104	4.2913	4.3030	4.3069	4.3103

0.2	0.5	5.5224	5.4890	5.4882	5.4634	5.4763	5.4824	5.4881
0.4	0.1	5.2123	5.2066	5.2479	5.2117	5.2104	5.2093	5.2076
0.4	0.3	9.0731	9.0695	9.0938	9.0848	9.0789	9.0750	9.0695
0.4	0.5	11.8009	11.7980	11.8171	11.8270	11.8153	11.8076	11.7977

σ	*T*	Tree			HSHP			Benchmark
		$\Delta_t = \frac{1}{960}$	$\Delta_t = \frac{1}{1920}$	$\Delta_t = \frac{1}{3840}$	$n = 6$	$n = 12$	$n = 24$	
RMSE		0.0325	0.0010	0.0173	0.0306	0.0165	0.0087	
RMSRE		0.0104	0.0003	0.0028	0.0044	0.0023	0.0012	
Time		0.0123	0.0432	0.1637	0.0317	0.0583	0.1028	

The parameters used are $S(0) = 50$ and $u(t) = S_{max}(t)/S(t) = 1.02$. $\Delta_t =$ 1/960, 1/1920, and 1/3840 which correspond to 4, 8, and 16 time steps per day in the binomial tree. The benchmark option values are evaluated according to Equation (9). The root-mean-squared absolute error (RMSE) is defined by, RMSE $= \sqrt{1/36 \sum_{i=1}^{36} e_i^2}$ where $e_i = |P_i^* - P_i|$ is the absolute error, P_i is the benchmark value, and P_i^* is the estimated option price using the tree method or HSHP approach. The root-mean-squared relative error (RMSRE) is defined by RMSRE $= \sqrt{1/36 \sum_{i=1}^{36} \hat{e}_i^2}$, where $\hat{e}_i = (P_i^* - P_i)/P_i$ is the relative error.

Semi-Static Hedging Portfolio

- The value of the *n*-point semi-static hedge portfolio (SSHP(n)) at time t can be rewritten as follows:

$$\mathrm{SSHP}(n) = P^E(S(t), S_{max}(t), \sigma, r, q, T - t) + u_{n-1}^* w_{n-1} P^E(S(t), S_{n-1}^*(t), \sigma, r, q, T - t)$$

$$+\widehat{w}_{n-1}C^E(S(t), S_{max}(t), \sigma, r, q, T - t)$$

$$+u_{n-2}^* w_{n-2} P^E(S(t), S_{n-2}^*(t), \sigma, r, q, t_{n-1} - t)$$

$$+\widehat{w}_{n-2}C^E(S(t), S_{max}(t), \sigma, r, q, t_{n-1} - t)$$

$$+ \cdots \ + u_0^* w_0 P^E(S(t), S_0^*(t), \sigma, r, q, t_1 - t)$$

$$+u_0^* w_0 P^E(S(t), S_0^*(t), \sigma, r, q, t_1 - t)$$

$$+\widehat{w}_0 C^E(S(t), S_{max}(t), \sigma, r, q, t_1 - t), \qquad (10)$$

where $S_i^*(t) = S_{max}(t)/u_i^*$. For a SSHP($n$), there are n vanilla put options and $n-1$ vanilla call options with different strike prices $S_i^*(t)$ and times to maturity $t_i - t$ for $i = 0,1,2,\ldots,n$.

- One special feature of SSHP
  - Different from traditional static hedging approaches, purchasing the corresponding SSHP(n) at issuance time point is not a once-and-for-all hedging strategy since the maximum of the stock price could change with the passage of time.

Hedging Performance of SSHP and DHP

The setting

- Choose the 15th contract in Table 1 for example.
- Simulate 2000 asset price paths with 1000 time steps per day.
- Examine the hedging performance of the SSHP approach with only $n = 6$, but for the DHP based on Babbs's (2000) model.
- Calculate delta values with 1000 time steps per day.
- For every m days, we rebalance the SSHP and DHP according to the information of the prevailing asset price and realized maximum asset price.

Risk measures

- We adopt four risk measures, suggested by Siven and Poulsen (2009), to evaluate the profit and loss distributions of the HE of the SSHP and DHP approaches.
- 5% value-at-risk: $VaR_{0.05} = inf\{z \in \Re; Pr(HE \geq z) \leq 0.05\}$
- Conditional value-at-risk: $ES_{0.05} = E[HE|HE \geq VAR_{0.05}]$
- Quadratic hedging error: $E[HE^2]$, where $HE^+ = max(0, HE)$
- The expected loss: $E[HE^+]$

Numerical results

- For different risk measures under $m = 1,2,4$, all risk measurement results in the SSHP approach are smaller than the counterparts in the DHP.
- As the number of days for rebalancing *m* becomes small, the improvements of the *HE* of the SSHP approach are much better than those of the DHP approach for all risk measures.
- It indicates that SSHP approach outperform the DHP approach in terms of generating smaller and more stable hedging errors.

Table 2. Hedging Performance of the SSHP and the DHP

Risk Measures	DHP	SSHP

	m = 4	m = 2	m = 1	m = 4	m = 2	m = 1
$VaR_{0.05}$	1.4005	1.1455	0.9414	0.6526	0.3834	0.2605
$ES_{0.05}$	2.1191	1.7057	1.3492	0.9462	0.5738	0.4095
$E[HE^2]$	0.8241	0.5247	0.2771	0.3028	0.1293	0.0375
$E[HE^+]$	0.7365	0.5817	0.4466	0.2971	0.1968	0.1143

This table compares the hedging error distributions for the SSHP ($n = 6$) and DHP approaches. Denote m as the time interval (measured in days) to rebalance the portfolio. The time to maturity of the American floating strike lookback put options put is half a year and the other parameters are as follows: $S(t) = 50$, $u(t) = S_{max}(t)/S(t) = 1.02$, $r = 0.05$, $q = 0$, and $\sigma = 0.2$. We simulate 2,000 stock price paths, with 1000 time steps per day. We used the four risk measures in Siven and Poulsen (2009) to evaluate the hedging performance.

Conclusion

- We extend the static hedging approach to price and hedge American floating strike lookback options under the Black–Scholes model.

- We first obtain the American floating strike lookback option values by constructing a hypothetic static hedging portfolio to match the complicated boundary conditions and terminal condition.

- Compared with the tree method, the recalculation of the HSHP for various American floating strike lookback options is easier and faster.

- The numerical accuracy of HSHP is generally superior to the tree method of Babbs (2000) and it exhibits the pattern of monotonic convergence.

- We replicate a American floating lookback options by transforming the HSHP into a SSHP.
- Finally, we also compare the hedging performance of SSHP versus DHP for an American floating strike lookback put option**.** The numerical analyses indicate that the hedging performance of the SSHP is far more stable than that of a delta-hedged portfolio with daily rebalance.

Thanks for your attention!

Comments and suggestions

are welcome!