A Unified Framework for Option Pricing under Regime Switching Models

Ning Cai

Department of Industrial Engineering and Logistics Management
The Hong Kong University of Science and Technology

Joint work with Steven Kou and Yingda Song

Motivations and Literature Review

- Regime changes are prevalent in financial markets.

Motivations and Literature Review

- Regime changes are prevalent in financial markets.
- Various regime switching (RS) models have been proposed.
  - See, e.g., Hamilton (1989, Econometrica) and the survey by Ang and Timmermann (2012, ARFE).

Motivations and Literature Review

- Mathematically intractable except special cases.

Motivations and Literature Review

- Mathematically intractable except special cases.
  - Special models
    - The RS-GBM model and the RS-JD model with exponential-type jumps

Motivations and Literature Review

- Mathematically intractable except special cases.
  - Special models
    - The RS-GBM model and the RS-JD model with exponential-type jumps
  - European options or first-passage-time related options
    - Barrier, lookback, and perpetual American options

Motivations and Literature Review

- Mathematically intractable except special cases.
  - Special models
    - The RS-GBM model and the RS-JD model with exponential-type jumps
  - European options or first-passage-time related options
    - Barrier, lookback, and perpetual American options
  - Continuously monitored options

Motivations and Literature Review

- Mathematically intractable except special cases.
  - Special models
    - The RS-GBM model and the RS-JD model with exponential-type jumps
  - European options or first-passage-time related options
    - Barrier, lookback, and perpetual American options
  - Continuously monitored options
  - See, e.g., Naik (1993), Di Masi et al. (1994), Bollen (1998), Guo (2001), Zhang and Guo (2004), Elliott et al. (2005), Yao et al. (2006), Jobert and Rogers (2006), Boyle and Draviam (2007), Jiang and Pistorius (2008), Siu et al. (2015), $\cdots$

Our Contributions

- We propose a unified analytical approximation approach to option pricing under regime switching models.

Our Contributions

- We propose a unified analytical approximation approach to option pricing under regime switching models.
  - Applies to general regime switching Markov models.

Our Contributions

- We propose a unified analytical approximation approach to option pricing under regime switching models.
  - Applies to general regime switching Markov models.
    - Model types in different regimes could be different.

Our Contributions

- We propose a unified analytical approximation approach to option pricing under regime switching models.
  - Applies to general regime switching Markov models.
    - Model types in different regimes could be different.
  - Applies to European, barrier, lookback, and Asian options.

Our Contributions

- We propose a unified analytical approximation approach to option pricing under regime switching models.
  - Applies to general regime switching Markov models.
    - Model types in different regimes could be different.
  - Applies to European, barrier, lookback, and Asian options.
  - Applies to both continuously and discretely monitored options.

Our Contributions

- We propose a unified analytical approximation approach to option pricing under regime switching models.
  - Applies to general regime switching Markov models.
    - Model types in different regimes could be different.
  - Applies to European, barrier, lookback, and Asian options.
  - Applies to both continuously and discretely monitored options.
- Potentially applicable to more finance problems.

Our Contributions

- We propose a unified analytical approximation approach to option pricing under regime switching models.
  - Applies to general regime switching Markov models.
    - Model types in different regimes could be different.
  - Applies to European, barrier, lookback, and Asian options.
  - Applies to both continuously and discretely monitored options.
- Potentially applicable to more finance problems.
  - Application I: Structural models of credit risk

Our Contributions

- We propose a unified analytical approximation approach to option pricing under regime switching models.
  - Applies to general regime switching Markov models.
    - Model types in different regimes could be different.
  - Applies to European, barrier, lookback, and Asian options.
  - Applies to both continuously and discretely monitored options.
- Potentially applicable to more finance problems.
  - Application I: Structural models of credit risk
  - Application II: Value of marketability

The Model

- Assume $\{S_t\}$ follows a general regime switching Markov models.

The Model

- Assume $\{S_t\}$ follows a general regime switching Markov models.
- The CTMC $\{Z_t\}$ accounts for regime switching with finite state space $\mathcal{S}_Z = \{1, 2, \cdots, R\}$ and transition rate matrix $\mathbf{\Lambda} = (\lambda_{ij})_{R \times R}$.

The Model

- Assume $\{S_t\}$ follows a general regime switching Markov models.
- The CTMC $\{Z_t\}$ accounts for regime switching with finite state space $\mathcal{S}_Z = \{1, 2, \cdots, R\}$ and transition rate matrix $\mathbf{\Lambda} = (\lambda_{ij})_{R \times R}$.
- In regime i, $\{S_t\}$ follows a general Markov process.

The Model

- Assume $\{S_t\}$ follows a general regime switching Markov models.
- The CTMC $\{Z_t\}$ accounts for regime switching with finite state space $\mathcal{S}_Z = \{1, 2, \cdots, R\}$ and transition rate matrix $\mathbf{\Lambda} = (\lambda_{ij})_{R\times R}$.
- In regime i, $\{S_t\}$ follows a general Markov process.
  - Model types in different regimes could be different.

The Model

- Assume $\{S_t\}$ follows a general regime switching Markov models.
- The CTMC $\{Z_t\}$ accounts for regime switching with finite state space $\mathcal{S}_Z = \{1, 2, \cdots, R\}$ and transition rate matrix $\mathbf{\Lambda} = (\lambda_{ij})_{R \times R}$.
- In regime i, $\{S_t\}$ follows a general Markov process.
  - Model types in different regimes could be different.
- $\{S_t, Z_t\}$ is a two-dimensional Markov process.

Key ideas

- A unified two-step analytical approximation approach

Key ideas

- A unified two-step analytical approximation approach
- **Step 1:** Construct a regime switching CTMC sufficiently close to the target regime switching Markov model;

Key ideas

- A unified two-step analytical approximation approach
- **Step 1:** Construct a regime switching CTMC sufficiently close to the target regime switching Markov model;
- **Step 2.** Derive analytical pricing formulas under the approximate regime switching CTMC.

Theorem 1 (Converting a Regime Switching CTMC to a one-dimensional CTMC)

Consider a regime switching CTMC $\{S_t\}$ that in regime i, follows a CTMC with state space $\mathcal{S} = \{x_1, x_2, \cdots, x_N\}$ and transition rate matrix $\boldsymbol{G}_i$ for $i = 1, \cdots, R$. Consider a one-dimensional CTMC $\{Y_t\}$ with state space $\mathcal{S}_Y = \{1, \cdots, N \cdot R\}$ and transition rate matrix

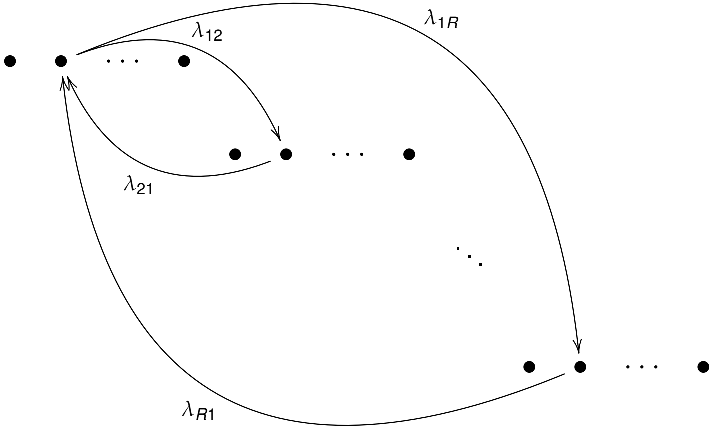
$$\boldsymbol{G} = \begin{pmatrix} \lambda_{11}\boldsymbol{I}_N + \boldsymbol{G}_1 & \lambda_{12}\boldsymbol{I}_N & \cdots & \lambda_{1R}\boldsymbol{I}_N \\ \lambda_{21}\boldsymbol{I}_N & \lambda_{22}\boldsymbol{I}_N + \boldsymbol{G}_2 & \cdots & \lambda_{2R}\boldsymbol{I}_N \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{R1}\boldsymbol{I}_N & \lambda_{R2}\boldsymbol{I}_N & \cdots & \lambda_{R \cdot R}\boldsymbol{I}_N + \boldsymbol{G}_R \end{pmatrix}.$$

Then we have

$$\mathbb{E}[\psi(S)|Z_0 = i, S_0 = x_k] = \mathbb{E}[\psi \circ \pi(Y)|Y_0 = (i-1)N + k].$$

Here $\pi : \mathcal{S}_Y \to \mathcal{S}$ with $\pi(n) := x_k$, where $k \in \mathcal{S}$ is the unique integer such that $n = (i-1)N + k$ for some $i \in \mathcal{S}_Z$.

Convert a regime switching CTMC to a one-dimensional CTMC.



European Options

Proposition 1 (European Options under Regime Switching CTMCs)

Given $Z_0 = i$ and $S_0 = x$, the price of a European call option at time 0 with maturity date T and strike price K is

$$\mathbb{E}\left[e^{-rT}(S_T - K)^+\right] = \boldsymbol{e}_{i,x_k} \cdot \mathrm{Exp}((\boldsymbol{G} - r\boldsymbol{I})T) \cdot \boldsymbol{K},$$

where $\boldsymbol{K}$ is an $N \cdot R \times 1$ vector with $K_{(i-1)N+j} = (x_j - K)^+$.

Numerical Examples

Prices and deltas of European options under the regime switching CEV model

	European Call Option Prices				European Call Option Δ			
K	RSCEV	CEV1	CEV2	DL	RSCEV	CEV1	CEV2	DL
95	10.6900	9.8687	12.6627	12.6629	0.8498	0.9266	0.7294	0.7292
100	7.0465	5.8502	9.5842	9.5845	0.7094	0.7684	0.6282	0.6282
105	4.2303	2.8621	7.0165	7.0170	0.5183	0.5178	0.5204	0.5202

Numerical Examples

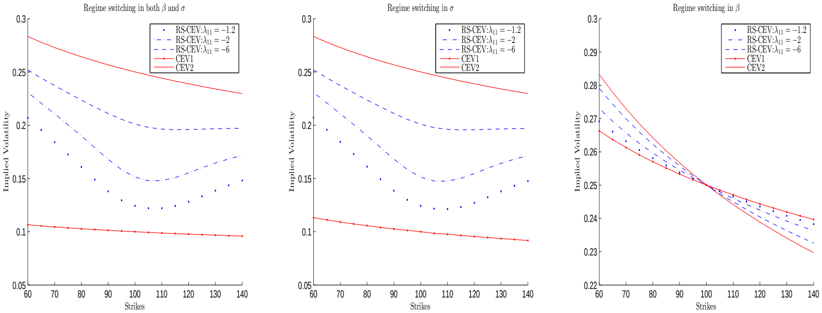


Figure: *Implied Volatilities under Regime Switching CEV Models.*

Consider a barrier (call) option, which knocks out when the underlying asset price exits U before maturity time.

Proposition 2 (Barrier Options under Regime Switching CTMCs)

Let $\boldsymbol{K}$ be an $N \cdot R \times 1$ vector with $K_{(i-1)N+j} = (x_j - K)^+$. Given $Z_0 = i$ and $S_0 = x$, the barrier option price at time 0 is

$$\mathbb{E}\left[e^{-rT}(S_T - K)^+ I_{\{\tau_U > T\}}\right]$$
$$=(\boldsymbol{e}_{i,x_k})_{[x\in U]}\mathrm{Exp}\left((\boldsymbol{G} - r\boldsymbol{I})_{[x\in U]}T\right)\boldsymbol{K}_{[x\in U]}.$$

Prices and deltas of various barrier options under the regime switching CEV model

K	RSCEV	CEV1	CEV2	DL	RSCEV	CEV1	CEV2	DL
	Down and Out Call Prices				Down and Out Call Δ			
95	10.2377	9.8447	10.6013	10.6013	0.9263	0.9389	0.9800	0.9800
100	6.8140	5.8462	8.3041	8.3042	0.7502	0.7708	0.7982	0.7982
105	4.1141	2.8613	6.2553	6.2554	0.5395	0.5181	0.6301	0.6300
	Up and Out Call Prices				Up and Out Call Δ			
95	6.7123	8.8021	3.1379	3.1383	0.2734	0.5787	-0.0438	-0.0439
100	3.8035	4.9877	1.7256	1.7260	0.2380	0.4867	-0.0190	-0.0190
105	1.7112	2.2036	0.7731	0.7734	0.1499	0.3019	-0.0064	-0.0066
	Capped Call Prices				Capped Call Δ			
95	10.4753	9.8325	11.8874	11.8877	0.8127	0.9122	0.6383	0.6383
100	6.8140	5.8120	8.7252	8.7256	0.6694	0.7534	0.5268	0.5267
105	3.9690	2.8218	6.0228	6.0231	0.4734	0.5020	0.4029	0.4028
	Double Barrier Call Prices				Double Barrier Call Δ			
95	6.3525	8.7782	1.8804	1.8805	0.3318	0.5912	0.0787	0.0787
100	3.6457	4.9839	1.0956	1.0958	0.2638	0.4890	0.0461	0.0461
105	1.6522	2.2032	0.5124	0.5126	0.1597	0.3022	0.0218	0.0217

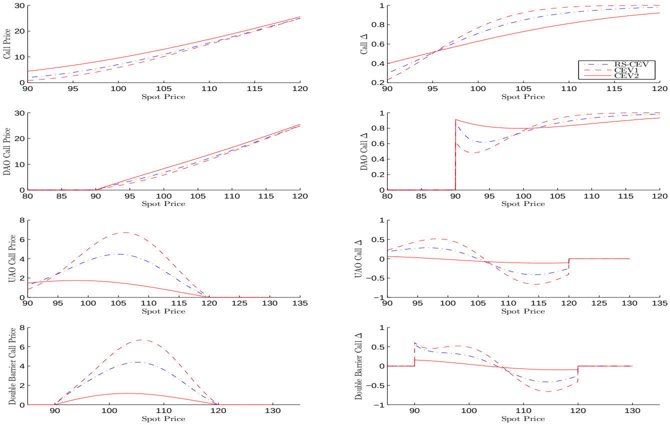


Figure: *Barrier Call Option Prices and Deltas under Regime Switching CEV Models.*

Proposition 4 (Lookback Options under Regime Switching CTMCs)

Let $M_t = \max_{0\le u\le t} S_t$, $m_t = \min_{0\le u\le t} S_t$. *Then given* $Z_0 = i$ *and* $S_0 = x$, *the prices of various lookback options are given by*

$$\mathbb{E}\left[e^{-rT}\left(S_T - \min\{m, m_T\}\right)\right] = e^{-rT}(V_1(m) - m) + x_k,$$

$$\mathbb{E}\left[e^{-rT}\left(\max\{M, M_T\} - S_T\right)\right] = e^{-rT}(V_2(M) + M) - x_k,$$

$$\mathbb{E}\left[e^{-rT}\left(\max\{M, M_T\} - K\right)^+\right] = \begin{cases} e^{-rT}V_2(K), & M \le K, \\ e^{-rT}(V_2(M) + M - K), & M > K, \end{cases}$$

$$\mathbb{E}\left[e^{-rT}\left(K - \min\{m, m_T\}\right)^+\right] = \begin{cases} e^{-rT}V_1(K), & m \ge K, \\ e^{-rT}(V_1(m) - m + K), & m < K. \end{cases}$$

Prices and deltas of various lookback options under the regime switching CEV model

K	RSCEV	CEV1	CEV2	DL	RSCEV	CEV1	CEV2	DL
	Standard Lookback Call Prices				Standard Lookback Call Δ			
N.A.	10.5870	8.2804	15.8810	15.8791	0.0762	0.0706	0.0956	0.0955
	Call on Maximum Prices				Call on Maximum Δ			
100	11.3197	8.4567	16.6138	16.6083	1.0289	1.0202	1.0446	1.0466
105	7.0642	4.3320	12.2641	12.2587	0.8095	0.7421	0.8800	0.8800
	Standard Lookback Put Prices				Standard Lookback Put Δ			
N.A.	6.4426	3.5796	11.7367	11.7312	0.0310	0.0227	0.0466	0.0466
	Put on Minimum Prices				Put on Minimum Δ			
95	2.5133	0.7071	6.9362	6.9342	-0.3864	-0.2369	-0.6384	-0.6383
100	5.7099	3.4033	11.0039	11.0021	-0.9136	-0.9156	-0.9008	-0.9045

Asian Options

Consider the Asian call option whose price at time 0 is

$$V_a(T,K) = e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right].$$

Proposition 5 (Asian Options under Regime Switching CTMCs)

Suppose that $S_0 = x_k$ and $Z_0 = i$. For any complex θ such that $\Re(\theta) > 0$, the Laplace transform of $V_a(T,K)$ w.r.t. K is given by

$$\begin{aligned}&\int_0^{+\infty} e^{-\theta K} V_a(t,K)dK \\ =& e^{-rT}\left(\frac{x_k(e^{rT}-1)}{\theta r T} - \frac{1}{\theta^2} + \frac{\mathbf{e}_{i,x_k}\cdot \mathrm{Exp}(T\mathbf{G}-\theta\mathbf{D})\cdot\mathbf{1}}{\theta^2}\right).\end{aligned}$$

Prices of Asian options under the regime switching CEV model

K	RS-CTMC	Simulation	Std. Err.	Abs. Err.	Rel. Err.	CPU Time (sec.)
90	12.380	12.367	0.009	0.013	0.10%	1.5
100	5.210	5.198	0.007	0.012	0.23%	1.4
110	1.491	1.482	0.004	0.009	0.62%	1.4

- Consider a bond under the Black-Cox model.

Application I: Structural Models of Credit Risk

- Consider a bond under the Black-Cox model.
- Define the default time $\tau := \inf\{t \geq 0 : S_t \leq K\}$.

- Consider a bond under the Black-Cox model.
- Define the default time $\tau := \inf\{t \geq 0 : S_t \leq K\}$.
- If $\tau > T$, the bond holder receives the face value.

Application I: Structural Models of Credit Risk

- Consider a bond under the Black-Cox model.
- Define the default time $\tau := \inf\{t \geq 0 : S_t \leq K\}$.
- If $\tau > T$, the bond holder receives the face value.
- If $\tau \leq T$, the bond holder receives $1 - \alpha(S_\tau)$ times the face value at τ.

Application I: Structural Models of Credit Risk

- Consider a bond under the Black-Cox model.
- Define the default time $\tau := \inf\{t \geq 0 : S_t \leq K\}$.
- If $\tau > T$, the bond holder receives the face value.
- If $\tau \leq T$, the bond holder receives $1 - \alpha(S_\tau)$ times the face value at τ.
- The bond value with unit face value at time 0 is given by

$$B(T) = \mathbb{E}\left[e^{-r\tau}(1 - \alpha(S_\tau))1_{\{\tau \leq T\}}\right] + e^{-rT}\mathbb{E}[1_{\{\tau > T\}}].$$

Application I: Structural Models of Credit Risk

Proposition 6 (Bond Prices and Instantaneous Yield Spreads under Regime Switching CTMCs)

Suppose that $Z_0 = i$ and $S_0 = x_k$. Then

$$B(T) = \boldsymbol{e}_{i,x_k} \cdot \mathrm{Exp}\left((\boldsymbol{G} - r\boldsymbol{I})_{\{x>K\}} T \right) \cdot \boldsymbol{w},$$

where $\boldsymbol{w}$ is an $N \cdot R \times 1$ vector with $w_{(i-1)N+j} = 1 - \alpha(x_j) 1_{\{x_j \leq K\}}$. In addition, we have

$$Y(0) := \lim_{T \to 0} -\frac{\log(B(T))}{T} - r = Y_k(0),$$

where Y_k is defined as the bond yield spread when the firm value is governed by the one-dimensional Markov process of the k-th regime.

Consider a regime switching double exponential jump diffusion model.

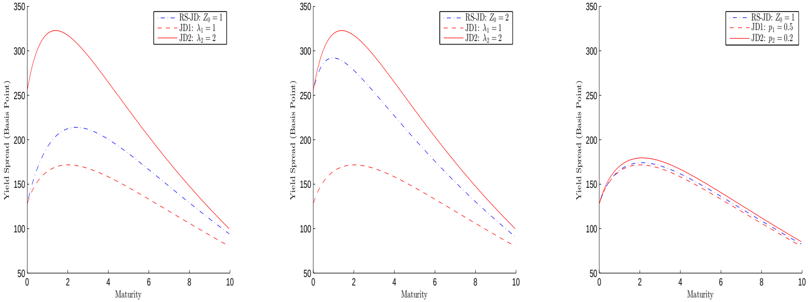


Figure: *The yield spreads under a two-state regime switching DEJD model and two corresponding DEJD models.*

Application II: Value of Marketability

- Let $\{S_t : 0 \leq t \leq T\}$ be the price of the underlying security traded in a frictionless market.

Application II: Value of Marketability

- Let $\{S_t : 0 \leq t \leq T\}$ be the price of the underlying security traded in a frictionless market.

- If restricted from trading the security before T, then the security is worth S_T to the investor (holding it) at time T.

Application II: Value of Marketability

- Let $\{S_t : 0 \leq t \leq T\}$ be the price of the underlying security traded in a frictionless market.

- If restricted from trading the security before T, then the security is worth S_T to the investor (holding it) at time T.

- Without restriction, the optimal value the investor could obtain at time T is $\max_{0 \leq t \leq T} \left(e^{r(T-t)} S_t \right)$.

Application II: Value of Marketability

- Let $\{S_t : 0 \leq t \leq T\}$ be the price of the underlying security traded in a frictionless market.
- If restricted from trading the security before T, then the security is worth S_T to the investor (holding it) at time T.
- Without restriction, the optimal value the investor could obtain at time T is $\max_{0\leq t\leq T}\left(e^{r(T-t)}S_t\right)$.
- Like Longstaff (1995, JF), an upper bound of the value of marketability is

$$\mathrm{UB} = e^{-rT}\mathbb{E}\left[\max_{0\leq t\leq T}\left(e^{r(T-t)}S_t\right) - S_T\right].$$

Application II: Value of Marketability

- Surprisingly, Longstaff (1995) finds that empirical evidence suggests that the upper bound is very tight.

Application II: Value of Marketability

- Surprisingly, Longstaff (1995) finds that empirical evidence suggests that the upper bound is very tight.
- The model used by Longstaff (1995) is the Black-Scholes model.

Application II: Value of Marketability

- Surprisingly, Longstaff (1995) finds that empirical evidence suggests that the upper bound is very tight.
- The model used by Longstaff (1995) is the Black-Scholes model.
- However, the Black-Scholes model is too simple to be realistic.

Application II: Value of Marketability

- Surprisingly, Longstaff (1995) finds that empirical evidence suggests that the upper bound is very tight.
- The model used by Longstaff (1995) is the Black-Scholes model.
- However, the Black-Scholes model is too simple to be realistic.
- Consider a regime switching lognormal jump diffusion model instead.

- Define $\tilde{S}_t := e^{-rt} S_t$. Then the dynamic of $\{\tilde{S}_t\}$ is the same as that of $\{S_t\}$ with $r = 0$.

Application II: Value of Marketability

- Define $\tilde{S}_t := e^{-rt} S_t$. Then the dynamic of $\{\tilde{S}_t\}$ is the same as that of $\{S_t\}$ with $r = 0$.
- The upper bound UB is equal to the price of a standard lookback put option contingent on $\{\tilde{S}_t\}$.

Application II: Value of Marketability

- Define $\tilde{S}_t := e^{-rt} S_t$. Then the dynamic of $\{\tilde{S}_t\}$ is the same as that of $\{S_t\}$ with $r = 0$.

- The upper bound UB is equal to the price of a standard lookback put option contingent on $\{\tilde{S}_t\}$.

- Define the percentage discount for lack of marketability as

$$\frac{\mathrm{UB}}{\mathbb{E}\left[\max_{0 \le t \le T} \tilde{S}_t\right]} = \frac{\mathbb{E}\left[\max_{0 \le t \le T} \tilde{S}_t\right] - S_0}{\mathbb{E}\left[\max_{0 \le t \le T} \tilde{S}_t\right]}.$$

Comparison with Longstaff's results

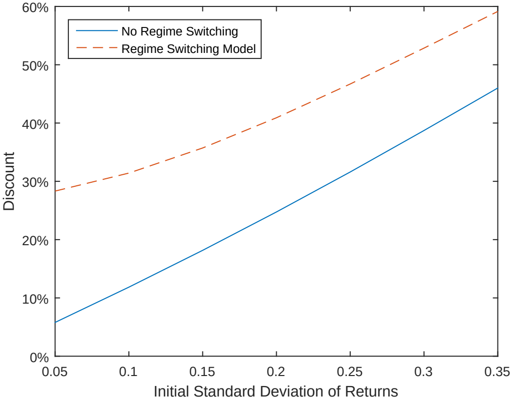


Figure: *Upper bounds for percentage discounts for lack of marketability vs. the initial standard deviation of returns.*

- Thank you!