Approximate Arbitrage-Free Option Pricing under the SABR Model

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Outline

The Stochastic-Alpha-Beta-Rho (SABR) Model

"Managing Smile Risk": Stochastic-Alpha-Beta-Rho Model
Hagan, Kumar, Lesniswski & Woodward (2002)

$$\begin{cases} dF_t = A_t F_t^{\beta} dW_t^1, \\ dA_t = \nu A_t dW_t, \end{cases}$$

where $\beta \in [0,1)$, $\nu > 0$, $|\rho| < 1$, $F_0 > 0$, $A_0 > 0$, and $< dW_t, dW_t^1 >= \rho dt$.

Description of the SABR Model

- Origins of the name: “stochastic” - stochastic volatility model, “alpha”-initial volatility (volatility process), “beta”- the index of the CEV component “rho”- the correlation between the volatility and underlying asset processes
- The underlying F can be forward price or forward rate.
- The SABR model is a generalization of the local volatility model with a stochastic volatility.
- The system of the SDEs is defined under the forward martingale measure.
- The case $\beta = 1$ is not considered because F is always positive. There is no arbitrage problem!

Implied Volatility Formula for the SABR Model

- A simplified version of closed-form asymptotic formula for implied volatility (formula (3.1) in Hagan et.al. (2002))

$$\sigma_{sabr}(K, F_0) = \frac{A_0}{F_0^{1-\beta}} \left\{ 1 + a \cdot \ln(\frac{K}{F_0}) + b \cdot \ln^2(\frac{K}{F_0}) + \ldots \right\}$$

$$\begin{cases} a = -\frac{1}{2}(1 - \beta - \rho\lambda), \qquad \lambda = \nu F_0^{1-\beta}/A_0 \\ b = \frac{1}{12}[(1-\beta)^2 + (2 - 3\rho^2)\lambda^2]. \end{cases}$$

- A quadratic function of logmoneyness. This simplified version of the implied volatility formula can help us explain why it can fit the volatility smile.
- Clear interpretation of model parameters
  - A_0: controls the overall height of the curve, $\sigma(K, K) = \frac{A_0}{F_0^{1-\beta}}$
  - ρ: controls the curve's skew
  - ν: controls the curve's smile

Implied Volatility Formula for the SABR Model (cont'd)

- Using a singular perturbation approach, Hagan et.al. (2002) derive the closed-form asymptotic formula for implied volatility
- A full version of the formulap

$$\sigma(K,F_0) = \frac{A_0}{(F_0K)^{(1-\beta)/2}\left\{1+\frac{(1-\beta)^2}{24}\ln^2\frac{F_0}{K}+\frac{(1-\beta)^4}{1920}\ln^4\frac{F_0}{K}\right\}+\cdots}\cdot\frac{z}{x(z)}$$

$$\cdot\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{A_0^2}{(F_0K)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu A_0}{(F_0K)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right]t_{ex}+\cdots\right\}$$

$$z=\frac{\nu}{A_0}(F_0K)^{(1-\beta)/2}\ln\frac{F_0}{K}, \qquad x(z)=\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}$$

- t_{ex} is the exercise date. Substituting this implied volatility formula into the Black's formula, we have the European option price.

Advantages of the SABR Model

- Analytical tractability: implied vol formula
    - The closed-form formula leads to fast pricing, hedging, and calibration.
    - The SABR model is used for volatility modeling, and thus calibration is an important issue in practice.
    - Calibration using finite difference method is slow and unimaginable! Therefore, the closed-form formula is critical.
- Fitting the markets well: volatility skews and smiles
    - Volatility skews are smiles are stylized facts in option markets.
    - The SABR model can fit the volatility smiles or skews.
    - Intuitively, this point can be understood from the quadratic form in the simplified version of implied volatility formula.

Advantages of the SABR Model (cont'd)

- Capturing the correct comovement dynamics of the implied volatility curves, which results in stable hedge (Hagan et al. (2002)) $\frac{\partial V_{call,sabr}}{\partial F} = \frac{\partial BS}{\partial F} + \frac{\partial BS}{\partial \sigma}\frac{\partial \sigma_{sabr}(K,F)}{\partial F}$
  - The local volatility model can also fit the market very well, but fail to capture the correct comovement.
  - The comovement is captured by the sign of the term in red.
  - An opposite sign will result in wrong hedge. Then, a deal may experience losses.
- Widely used by practitioners in interest rate and foreign exchange markets

Literatures about Implied Volatility Formula

- Hagan et al.(2002): asymptotic implied volatility formula via singular perturbation approach
- Henry-Labordere (2005): asymptotic implied volatility at the first-order for any stochastic volatility model
- Osajima (2007): second order asymptotic expansion of implied volatility for the dynamic SABR model via Malliavin calculus approach
- Obłój(2008): correcting Hagan et al.(2002)
- Paulot (2015): first order analytical asymptotic expansion of implied volatility for SABR model
- ...

Arbitrage Problem in the SABR Model

- Forward price F can hit zero with POSITIVE probability due to the CEV-structure!
  - A CEV process
  $$dF_t = F_t^{\beta} dW_t$$
  - Feller's test of explosion: if $\beta \leq \frac{1}{2}$, a boundary condition at zero should be specified to determine the uniqueness of the solution to the SDE.
  - A reflecting or absorbing boundary should be specified at zero. In other words, F can hit zero with positive probability.
- For the SABR model, an absorbing boundary at $F = 0$ has to be specified to avoid the arbitrage opportunity.
  - If it is a reflecting boundary at $F = 0$, then we buy the forward once it hits zero, and sells it out when it is reflected to the positive value.
  - The probability of reflecting back is positive, thus there is an arbitrage opportunity for a reflecting boundary at $F = 0$.

Arbitrage Problem in the SABR Model (cont'd)

- The low rate environment after the crisis amplifies the probability of hitting zero because the initial forward rate F_0 is approaching zero.
- The implied volatility formulas developed by Hagan et al. (2002), Obłój (2008), Paulot (2015) do not account for the probability of hitting zero.
- Hence their implied volatility formulas suffer from inaccuracies due to the arbitrage problem.
- As the strike price decreases, the impact of the probability of hitting zero is getting larger. Thus, for the option with low strikes, this arbitrage problem becomes more significant.
- To avoid arbitrage, the absorbing boundary at zero should be considered.

Motivation

- Assumption: 0 is an absorbing boundary of the underlying process F.
- Pricing an European call without arbitrage $\Leftrightarrow$ pricing a down-and-out call with 0 as the knock-out boundary.
- **Question: Can we find an approximate solution to the arbitrage-free European option pricing problem?**
    - Practically significant: closed-form approximation can lead to fast pricing, hedging, and calibration.
    - Mathematically challenging: closed-form approximation formula for a continuously monitored barrier option price under a nontrivial two-dimensional model is notoriously difficult.

Literature Review: Arbitrage-Free Option Pricing of the SABR Model

- Doust (2012) proposes a remedy by numerically computing the probability that the forward price hits zero.
- Balland and Tran (2013) present a method based on normal volatility expansion with absorption at zero in a special case (i.e., $\beta = 0$)
- Hagan et al. (2014) develop an algorithm to numerically solve a reduced PDE, which is one-dimensional and pobtained via asymptotic technique.
- Gulisashvili et al. (2014), Yang and Wan (2016) study the mass at zero, and apply it to small strike implied volatility.

None of the above literature provides **(approximate) analytical formula** to approximate the arbitrage-free European option price!

Literature Review: Pricing Barrier Options Analytically

- The symmetry of the model do matter in pricing barrier options analytically!
- One-dimensional models
  - " Reflection principle " or "method of image" leads to the analytical formula for the barrier option under the Black-Scholes model.
  - Davydov & Linetsky (2001, 2004), Linetsky (2004), Linetsky (2008) solve the barrier option pricing problems analytically under different one dimensional diffusion processes because their infinitesimal generators are at least formally self-adjoint.
- Higher dimensional models
  - "Reflection principle" leads to the analytical formula for the barrier option under the multivariate geometric BM.
  - Under the fast mean-reverting stochastic volatility models, Ilhan et al. (2004) obtain approximate solution to the barrier options by reducing the problem into a one-dimensional model.

Main Results

- Pricing barrier options analytically under stochastic volatility models is mathematically challenging, so is the arbitrage-free option pricing under the SABR model.
- Our contribution
  - We provide an approximate solution with explicit formula to the arbitrage-free European option pricing problem under the SABR model for the first time.
  - Numerical results show the accuracy of our formulas.

Formulation of the Arbitrage-Free Option Pricing Problem

- Let $\tau_0 = \{s \geq t : F_s = 0\}$ be the first time the underlying process F hits the lower boundary zero.
- Under the forward martingale measure, the option price is the expectation of its payoff function.
- Under the assumption that F does not hit zero before t, the arbitrage-free European option price is given by

$$V_c(t, f, a) = \mathbb{E}[(F_T - K)^+ \mathbf{1}_{\{\tau_0 > T\}} | F_t = f, A_t = a],$$

- $V_c(t, f, a)$ satisfies the following backward Kolmogorov equation (BKE)

$$\begin{cases} \frac{\partial V_c}{\partial t} + \frac{1}{2}[a^2 f^{2\beta} \frac{\partial^2 V_c}{\partial f^2} + 2\rho\nu a^2 f^{\beta} \frac{\partial^2 V_h}{\partial f \partial a} + \nu^2 a^2 \frac{\partial^2 V_h}{\partial a^2}] = 0 \\ V_c(t, 0, a) = 0 \\ V_c(T, f, a) = h(f). \end{cases} \quad (1)$$

Main Idea to Find the Approximate Solution

- Recalling the literatures, to find the analytical solutions of the barrier options is to find the symmetry property of the model.
- There are two analytical approaches to address IBVPs.
  - Spectral methods (eigenfunction expansion).
    For one-dimensional models, Davydov & Linetsky (2001, 2004), Linetsky (2004), etc. use it to find the analytical pricing formulas for barrier options.
  - Reflection principle or method of image
    Black-Scholes, multivariate geometric Brownian motion, etc.
- The SABR model
  - It is a two-dimensional local stochastic volatility model. To solve it analytically, we have to find the inherent symmetry of the SABR model.
  - We perform three-step operations to find the symmetry.

Operation I: Rescaling

- Consider the European call option, satisfying PDE (1)

$$V_c(t, f, a) = \mathbb{E}[(F_T - K)^+ \mathbf{1}_{\{\tau_0 > T\}} | F_t = f, A_t = a].$$

- Scaling: $t \to \tau$, $(f, a) \to (f, g)$

$$\tau = \frac{T - t}{T}, \quad f = f, \quad g = \frac{a}{\nu}.$$

- The option price under the coordinate (f, g):

$$P_1(\tau, f, g) := V_c(t, f, a) \equiv V_c(T(1 - \tau), f, \nu g)$$

- Parameter for expansion: $\epsilon = \nu\sqrt{T}$.
- BKE under new coordinates (f, g):

$$\begin{cases} \frac{\partial P_1}{\partial \tau} = \frac{\epsilon^2}{2}[g^2 f^{2\beta} \frac{\partial^2 P_1}{\partial f^2} + 2\rho g^2 f^{\beta} \frac{\partial^2 P_1}{\partial f \partial g} + g^2 \frac{\partial^2 P_1}{\partial g^2}], \\ P_1(\tau, 0, g) = 0, \\ P_1(0, f, g) = (f - K)^+. \end{cases}$$

Operation II: Lamperti-type transform

- The Lamperti-transform
  - A standard technique to turn a one-dimensional diffusion process into a new one with a volatility of unity
  - The one-dimensional diffusion is a drifted Brownian motion up to a Lamperti-transform. Thus, it has the symmetry property.
  - We find the inherent symmetry of the SABR model using a similar transformation.
- Lamperti-type transform: $(f, g) \to (x, y)$

$$\begin{cases} x = \int_{0}^{f} \frac{dx}{\epsilon x^{\beta} g} = \frac{f^{1-\beta}}{\epsilon(1-\beta)g}, & \beta \in [0,1) \\ y = g. \end{cases}$$

  - There is no arbitrage for $\beta = 1$ because F is always positive.
  - In this transformation, we map 0 into 0, which is a constant line instead of a curve depending on y or g.
  - This transformation stretches the coordinate near the boundary 0, which also makes it numerically feasible.

Operation II: Lamperti-type transform (cont'd)

- The option price under the coordinates (x, y):

$$P_2(\tau, x, y) := P_1(\tau, f, g) \equiv P_1(\tau, (\epsilon(1-\beta)xy)^{1/(\beta-1)}, y)$$

- Under new coordinates (x, y), $P_2(\tau, x, y)$ satisfies the following BKE:

$$\begin{cases} (\frac{\partial}{\partial \tau} - \frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{1-2\theta}{2x}\frac{\partial}{\partial x})P_2(\tau, x, y) = (\epsilon\rho\mathcal{L}_1 + \epsilon^2\mathcal{L}_2)P_2(\tau, x, y), \\ P_2(\tau, 0, y) = 0, \\ P_2(0, x, y) = ((\epsilon(1-\beta)xy)^{2\theta} - K)^+. \end{cases} \tag{2}$$

where $\theta = \frac{1}{2(1-\beta)}$, and

$$\begin{cases} \mathcal{L}_0 = \frac{\partial}{\partial \tau} - \frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{1-2\theta}{2x}\frac{\partial}{\partial x}, \\ p\mathcal{L}_1 = -2x\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} + y\frac{\partial^2}{\partial x \partial y}, \\ \mathcal{L}_2 = x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} + \frac{1}{2}y^2\frac{\partial^2}{\partial y^2} - xy\frac{\partial^2}{\partial x \partial y}. \end{cases}$$

Where is the Symmetry

- Recall $\mathcal{L}_0 = \frac{\partial}{\partial \tau} - \left(\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{1-2\theta}{2x} \frac{\partial}{\partial x} \right)$ and the absorbing boundary condition at 0.
- $\mathcal{L}_0$ is the infinitesimal generator of a $(2-2\theta)$-dimensional Bessel process with an absorbing boundary at 0.
- Here the symmetry comes. The Bessel process has the symmetry naturally because it is one-dimensional.
- Intuitively, if $\rho = \epsilon = 0$, we can solve (2). The initial-boundary value problem associated with $\mathcal{L}_0$ can be solved analytically.

Operation III: Homogenization

- The initial condition $((\epsilon(1-\beta)xy)^{2\theta} - K)^+$ in (2) contains ϵ.
- To remove ϵ from the terminal value, we need to perform a further operation:

$$\gamma(\epsilon)P(\tau, u, v) = P_2(\tau, f, g),\ \gamma(\epsilon) := (\epsilon(1-\beta))^{2\theta}.$$

- A change of strike price K as follows: $k = \frac{K^{1-\beta}}{\epsilon(1-\beta)}$.
- BKE for $P(\tau, x, y)$

$$\begin{cases} \mathcal{L}_0 P(\tau, x, y) = (\epsilon\rho\mathcal{L}_1 + \epsilon^2\mathcal{L}_2)P(\tau, x, y), \\ P(\tau, 0, y) = 0, \\ P(0, x, y) = ((xy)^{2\theta} - k^{2\theta})^+, \end{cases} \tag{3}$$

Solution for $\epsilon = 0$ and $\rho = 0$

Lemma

The solution for the equation

$$\left(\frac{\partial}{\partial \tau} - \frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{1-2\theta}{2x}\frac{\partial}{\partial x}\right) P(\tau, u) = f(\tau, x), \quad P(\tau, 0) = 0, \quad P(0, x) = g(x).$$

is given by the following formula:

$$P(\tau, x, y) = \int_0^{\tau} \int_0^{+\infty} \Gamma(\tau - s, x, \xi) f(\tau, \xi) d\xi ds + \int_0^{+\infty} \Gamma(\tau, x, \xi) g(\xi) d\xi,$$

where $\Gamma(\tau, u, \xi)$ *is given by*

$$\Gamma(\tau, x, \xi) = \frac{x^{\theta}\xi^{1-\theta}}{\tau} \exp\left(-\frac{x^2+\xi^2}{2\tau}\right) I_{\theta}\left(\frac{x\xi}{\tau}\right),$$

and $I_{\theta}(z)$ *is the modified Bessel function of the first kind given by*

$$I_{\theta}(z) = \sum_{n=0}^{+\infty} \frac{(z/2)^{2n+\theta}}{m!\Gamma(1+m+\theta)}.$$

An Expansion with Two Parameters: ρ and ϵ

- Expanding p$P(\tau, x, y)$ with respect to ϵ and ρ as follows:

$$P(\tau, x, y) = P^{(0)}(\tau, x, y) + \epsilon \cdot P^{(1,0)}(\tau, x, y) + \rho \cdot P^{(0,1)}(\tau, x, y) + O(\max(\epsilon^2, |\rho|\epsilon, \rho^2)).$$

- Substituting the above formula into the PDE (3), and comparing the coefficients of ϵ, ρ, then $P^{(0)}(\tau, x, y)$ satisfies

$$\begin{cases} \mathcal{L}_0 P^{(0)}(\tau, x, y) = 0, \\ P^{(0)}(\tau, 0, y) = 0, \\ P^{(0)}(0, x, y) = ((xy)^{2\theta} - k^{2\theta})^+. \end{cases} \tag{4}$$

$$\begin{cases} \mathcal{L}_0 C^{(1,0)}(\tau, x, y) = 0, \\ C^{(1,0)}(\tau, 0, y) = 0, \\ C^{(1,0)}(0, x, y) = 0; \end{cases} \qquad \begin{cases} \mathcal{L}_0 C^{(0,1)}(\tau, x, y) = 0, \\ C^{(0,1)}(\tau, 0, y) = 0, \\ C^{(0,1)}(0, x, y) = 0. \end{cases}$$

An Expansion with Two Parameters: ρ and ϵ (cont'd)

- The differential operator in (3) only consists of the leading order operator $\mathcal{L}_0$ and the second order operator $\epsilon\rho\mathcal{L}_1$ and $\epsilon^2\mathcal{L}_2$.
  - $P^{(1,0)}(\tau, x, y) = P^{(0,1)}(\tau, x, y) = 0$
  - the high order term only with respect to ρ are all zeros, that is, $P^{(0,m)}(\tau, x, y) = 0$ for $m \geq 2$.
- The expansion of $P(\tau, x, y)$ now reads

$$P(\tau, x, y) = P^{(0)}(\tau, x, y) + O(\max(\epsilon^2, \epsilon|\rho|)).$$

- No remainder term at the order of ρ^2!

The Explicit Formula for $P^{(0)}$

- Applying the Lemma to solve the PDE (4), we find that

$$P^{(0)}(\tau, x, y) = (xy)^{2\theta}(1-Q(\frac{k^2}{\tau y^2}; 2\theta+2, \frac{x^2}{\tau})) - k^{2\theta} Q(\frac{x^2}{\tau}; 2\theta, \frac{k^2}{\tau y^2})$$

where $Q(x; \kappa, y)$ given below is the cumulative distribution function of the noncentral chi-square distribution.

$$Q(x; \kappa, y) = \int_0^x \frac{1}{2} e^{-\frac{\xi+y}{2}} (\xi/y)^{(\kappa-2)/4} I_{(\kappa-2)/2}(\sqrt{\xi y}) d\xi,$$

- The error term: $\mathcal{R} \sim \mathcal{O}\left(\epsilon \cdot \max\{|\rho|, \epsilon\}\right)$.

Approximate Formula for the Call

- Noting the change of function and coordinates in each step, we can go back to find the approximate to $V_c p(t, f, a)$,

$$\bar{V}_c(t,f,a) = f \cdot \left(1 - Q\left(\frac{K^{2(1-\beta)}/(1-\beta)^2}{a^2(T-t)}; \frac{3-2\beta}{1-\beta}, \frac{f^{2(1-\beta)}/(1-\beta)^2}{a^2(T-t)}\right)\right) - K \cdot Q\left(\frac{f^{2(1-\beta)}/(1-\beta)^2}{a^2(T-t)}; \frac{1}{1-\beta}, \frac{K^{2(1-\beta)}/(1-\beta)^2}{a^2(T-t)}\right). \quad (5)$$

- The expansion of the call option price now reads as follows:

$$V_c(t,f,a) = \bar{V}_c(t,f,a) + O(\epsilon^{\frac{2-\beta}{1-\beta}} \cdot \max(\epsilon, |\rho|)).$$

Approximate Solutions to Arbitrage-Free Option Prices

Theorem

The approximate solution of the arbitrage free option problem (1) *is given below. For the call option, the approximate solution* $\bar{V}_c(t,f,a)$ *is*

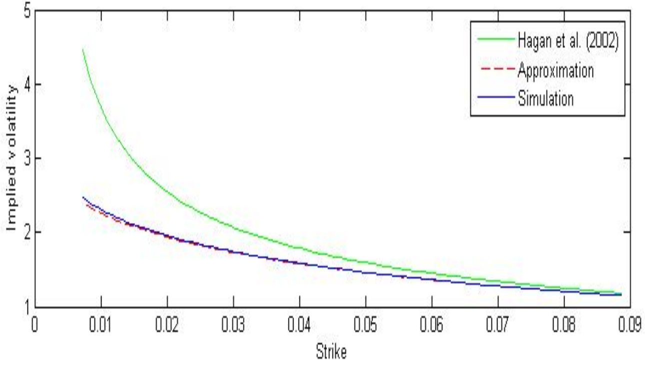
$$\bar{V}_c(t,f,a) = f \cdot \left(1 - Q\left(\frac{K^{2(1-\beta)}/(1-\beta)^2}{a^2(T-t)}; \frac{3-2\beta}{1-\beta}, \frac{f^{2(1-\beta)}/(1-\beta)^2}{a^2(T-t)}\right)\right) - K \cdot Q\left(\frac{f^{2(1-\beta)}/(1-\beta)^2}{a^2(T-t)}; \frac{1}{1-\beta}, \frac{K^{2(1-\beta)}/(1-\beta)^2}{a^2(T-t)}\right).$$

For the put option, the approximate solution $\bar{V}_p(t,f,a)$ *is*

$$\bar{V}_p(t,f,a) = K \cdot \left(1 - Q\left(\frac{f^{2(1-\beta)}/(1-\beta)^2}{a^2(T-t)}; \frac{1}{1-\beta}, \frac{K^{2(1-\beta)}/(1-\beta)^2}{a^2(T-t)}\right)\right) - f \cdot Q\left(\frac{K^{2(1-\beta)}/(1-\beta)^2}{a^2(T-t)}; \frac{3-2\beta}{1-\beta}, \frac{f^{2(1-\beta)}/(1-\beta)^2}{a^2(T-t)}\right);$$

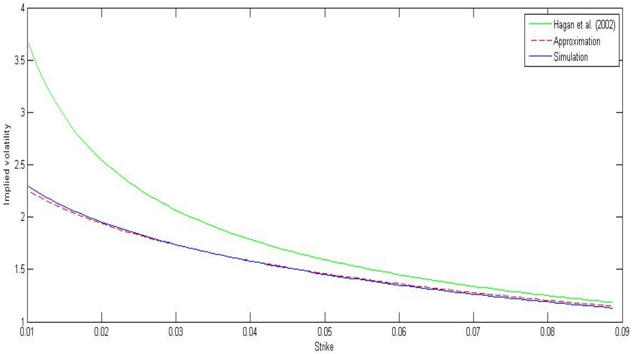
Comparison of Implied Volatilities: I

This picture plots the implied Volatilities generated by Hagan et al. (2002) (green), our approximate solution (red), and Monte Carlo simulation (blue). The parameters are given by $f = 0.05$, $a = 0.1$, $T = 1$, $\rho = 0$, $\nu = 0.1$, $\beta = 0.1$.



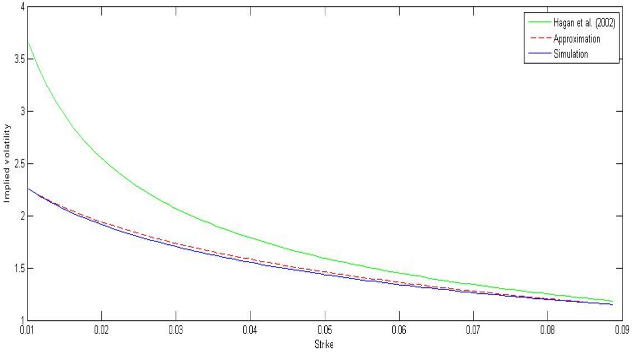
Comparison of Implied Volatilities: II

This picture plots the implied Volatilities generated by Hagan et al. (2002) (green), our approximate solution (red), and Monte Carlo simulation (blue). The parameters are given by $f = 0.05$, $a = 0.1$, $T = 1$, $\rho = -0.5$, $\nu = 0.1$, $\beta = 0.1$.



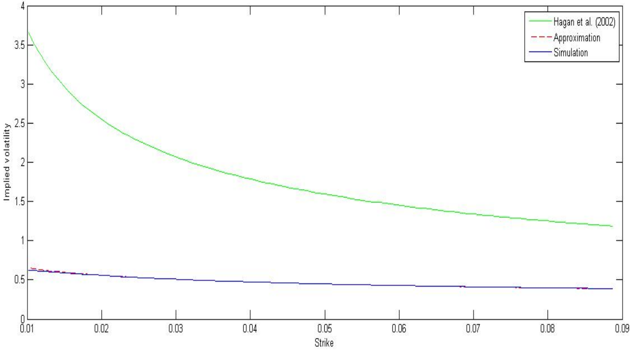
Comparison of Implied Volatilities: III

This picture plots the implied Volatilities generated by Hagan et al. (2002) (green), our approximate solution (red), and Monte Carlo simulation (blue). The parameters are given by $f = 0.05$, $a = 0.1$, $T = 1$, $\rho = 0$, $\nu = 0.5$, $\beta = 0.1$.



Comparison of Implied Volatilities: IV

This picture plots the implied Volatilities generated by Hagan et al. (2002) (green), our approximate solution (red), and Monte Carlo simulation (blue). The parameters are given by $f = 0.05$, $a = 0.1$, $T = 1$, $\rho = 0$, $\nu = 0.1$, $\beta = 0.5$.



Relative Errors of Approximate Option Prices under Different Maturities

T	1	5	10	15	20	25
Appr. Call	0.31%	0.27%	0.01%	0.46%	0.94%	0.92%
Appr. Put	0.96%	1.13%	1.28%	1.25%	1.40%	1.35%

Other parameters values are $\rho = -0.2$, $f = 0.05$, $a = 0.1$, $\beta = 0.1$, $\nu = 0.1$, and $K = 0.05$.

Relative Errors of Approximate Option Prices under Different Correlations

ρ	-0.1	-0.3	-0.5	-0.7	-0.9
Appr. Call	0.04%	0.30%	0.24%	0.74%	0.61%
Appr. Put	0.77%	0.98%	1.53%	1.62%	2.06%
ρ	0.1	0.3	0.5	0.7	0.9
Appr. Call	0.32%	0.39%	1.27%	1.36%	1.63%
Appr. Put	0.50%	0.15%	0.12%	0.12%	0.34%

Other parameters values are $T = 1$, $f = 0.05$, $a = 0.1$, $\beta = 0.1$, $\nu = 0.1$, and $K = 0.05$.

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Thanks for your attention!

Questions or comments?