VaR and CVaR Estimation for oil prices via SV-ALD model: A Bayesian approach using scale mixture of uniform distribution

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- Introduction
- Model framework
- Estimation methodology
- Empirical analysis
- Concluding remarks

Motivation

- Large fluctuations of oil prices.
  - Return uncertainty risks
  - Oil prices$\downarrow$ $\Rightarrow$ losses of supplier; Oil prices$\uparrow$ $\Rightarrow$ losses of demander
- Value-at-Risk(VaR)? A "coherent" risk measure: Conditional Value-at-Risk (CVaR) (Artzner et al., 1999).
- GARCH-type models? Stochastic volatility (SV) models' superiority in modelling volatility features (Regnier, 2007; Vo, 2009; Vo, 2011; Brooks and Prokopczuk, 2013; Wichitaksorn et al., 2014, etc.).
- Extension of SV-Normal (Meyer and Yu, 2000) to non-Gaussian distributions, i.e. SV-t (Chib et al., 2002; Asai, 2008; Abanto-Valle et al., 2010, etc), SV-SGT (Chai et al., 2011) and SV-generalized t (Wang et al., 2011).
  - Extending Asymmetric Laplace distribution (ALD) proposed by Kotz et al. (2001) to SV model.

Introduction

- A parametric approach to measure market risk of oil prices considering both of oil supplier and oil demander.
- Construction of SV-ALD model:
  - Standard discrete SV model to take account of time-varying volatility.
  - Extending return error term to be Asymmetric Laplace distributed.
- Analytical expression of VaR and CVaR for oil supplier and demander.
- Scale mixture of uniform (SMU) representation of ALD.
  - To facilitate an efficient Gibbs sampling algorithm in Bayesian MCMC.
- Backtesting risk measuring model (LR_{uc}, LR_{ind} and LR_{cc}).
  - A viable backtesting procedure of CVaR on the basis of "equal quantiles".

Stochastic volatility model

- A discrete SV model is employed shown as (Similar studies see Jian et al. (2011) and Chan et al. (2016)):

$$y_t = \mu + \sigma_t z_t \qquad z_t \sim ALD(\kappa, \tau, \theta) \quad (1)$$

$$ln\sigma_t^2 = h_t = \alpha + \beta(ln\sigma_{t-1}^2 - \alpha) + \eta_t \qquad \eta_t \sim N(0, \sigma_\eta^2) \quad (2)$$

  Where y_t is the asset return at time t, μ denotes mean of returns, h_t is the log-volatility at time t which follows a stationary AR(1) process with persistence parameter β having $|\beta| < 1$, z_t and η_t represent a series of i.i.d. random errors and uncorrelated.

- Conditional and unconditional distribution of h_t are expressed as:

$$h_t | h_{t-1}, \alpha, \beta, \sigma_\eta^2 \sim N(\alpha + \beta(h_{t-1} - \alpha), \sigma_\eta^2) \qquad t = 2, 3, .., T \quad (3)$$

$$h_t \sim N(\alpha, \frac{\sigma_\eta^2}{1 - \beta^2}) \qquad t = 1, 2, ..., T \quad (4)$$

Asymmetric Laplace density function

According to Kotz et al. (2001):

Definition

The random variable z is said to be distributed as ALD if its $p.d.f.(f(z|\kappa,\theta,\tau))$ can be demonstrated in the following form:

$$f(z|\kappa,\theta,\tau)=\begin{cases}\frac{\sqrt{2}}{\tau}\frac{\kappa}{1+\kappa^2}\ exp(-\frac{\sqrt{2}\kappa}{\tau}(z-\theta)) & z\geq\theta\\ \frac{\sqrt{2}}{\tau}\frac{\kappa}{1+\kappa^2}\ exp(\frac{\sqrt{2}}{\tau\kappa}(z-\theta)) & z<\theta\end{cases} \quad (5)$$

Where κ is skewness parameter, θ is location parameter, τ is scale parameter.

Risk of oil supplier

- **VaR**: Estimated VaR of oil supplier can be expressed as:

$$P(y_t \leq -VaR_{\mathrm{s,t}}|\Omega_t) = \int_{-\infty}^{-m_{s,t}} f^{-}(z_t)\, dy_t = \alpha \tag{6}$$

Where $m_{s,t}$ is set to be equal to $\frac{VaR_{s,t}-\mu}{\sigma_t}$, $f^{-}(y_t)$ is the negative part of $p.d.f.$ of ALD. Hence, the closed-form of VaR for oil supplier under SV-ALD model is given by:

$$VaR_{s,t} = \mu + m_{s,t}\,\sigma_t = \mu - \frac{\kappa\tau\sigma_t}{\sqrt{2}}\, ln \frac{\alpha(1+\kappa^2)}{\kappa^2} \tag{7}$$

- **CVaR**:

$$\begin{aligned} CVaR_{s,t} &= -\,E[y_t|y_t \leq -VaR_{s,t}] \\ &= -\,2\mu + VaR_{s,t} + \frac{\kappa\tau\sigma_t}{\sqrt{2}} \end{aligned} \tag{8}$$

  - Note that $CVaR_{s,t}$ is simply a function of α, κ, τ and σ_t.

Risk of oil demander

- **VaR**: Analogously, the estimated VaR of oil demander is written as:

$$P(y_t > VaR_{\mathrm{d,t}}|\Omega_t) = \int_{m_{d,t}}^{+\infty} f^+(z_t)\,dy_t = \alpha \tag{9}$$

Where $m_{d,t} = \frac{VaR_{d,t}-\mu}{\sigma_t}$, $f^+(y_t)$ is the positive part of $p.d.f.$ of ALD. Solving the integral, a closed-form VaR for oil demander under SV-ALD model can be written as:

$$VaR_{d,t} = \mu + m_{d,t}\sigma_t = \mu - \frac{\tau\sigma_t}{\sqrt{2}\kappa} ln(\alpha(1+\kappa^2)) \tag{10}$$

- **CVaR**:

$$\begin{aligned} CVaR_{d,t} =& E[y_t|y_t > VaR_{d,t}] \\ =& VaR_{d,t} + \frac{\tau\sigma_t}{\sqrt{2}\kappa} \end{aligned} \tag{11}$$

  - Likewise, $CVaR_{d,t}$ is a function of α, κ, τ and σ_t.

MLE

- MLE approach is used for the estimation of parameter κ and τ when return residuals are obtained. For MLE methodology of ALD to see Kotz et al. (2001).

Bayesian MCMC

- For model parameters $\dot{\theta} = (\alpha, \beta, \sigma_\eta^2)$ and latent volatilities $h = (h_1, ..., h_T)$, a simulation-based MCMC algorithm via Gibbs sampling scheme is used for Bayesian inference.
- Sampling from full conditional posterior distributions:

$$f(\alpha|\beta, \sigma_\eta^2, h, y) \propto f(y|\dot{\theta}, h) f(h|\dot{\theta}) f(\alpha) \quad (12)$$

$$f(\beta|\alpha, \sigma_\eta^2, h, y) \propto f(y|\dot{\theta}, h) f(h|\dot{\theta}) f(\beta) \quad (13)$$

$$f(\sigma_\eta^2|\alpha, \beta, h, y) \propto f(y|\dot{\theta}, h) f(h|\dot{\theta}) f(\sigma_\eta^2) \quad (14)$$

$$f(h_t|h_{-t}, \dot{\theta}, y) \propto f(y|h_t, \dot{\theta}) f(h_t|h_{-t}, \dot{\theta}) \quad (15)$$

- This procedure is realized by proposing the following SMU representation of ALD.

Scale mixture of uniform (SMU) representation

- Expressing ALD as an SMU helps to simplify the estimation method, especially in Bayesian analysis using Gibbs sampling algorithm (More to see Choy and Chan, 2008; Wichitaksorn et al., 2014).

Proposition

If $\lambda \sim Ga(2,1)$ *and* $z \sim U(\theta - \frac{\kappa\tau\lambda}{\sqrt{2}}, \theta + \frac{\tau\lambda}{\sqrt{2}\kappa})$*, then the SMU density:*

$$f(z|\kappa,\tau,\theta,\lambda) = \int_0^\infty f_U(z|\theta - \frac{\kappa\tau\lambda}{\sqrt{2}}, \theta + \frac{\tau\lambda}{\sqrt{2}\kappa}) \times f_{Ga}(\lambda|2,1)\, d\lambda \quad (16)$$

has the same form as that of the AL density function given in (5).

Where $f_{Ga}(c,d)$ *is the gamma density function of the form:*

$$Ga(\lambda|c,d) = \frac{1}{\Gamma(c)d^c}\lambda^{c-1}\, exp(-\frac{\lambda}{d}) \qquad \lambda, c, d > 0 \quad (17)$$

with shape parameter c *and scale parameter* d *and* $\Gamma(c)$ *is gamma function evaluated at* c*.*

- A scaled ALD (SALD) is used when taking into account of time-varying volatilities. ALD random variable is scaled by its standard deviation (Also refer to Chen et al.(2009) and Wichitaksorn et al.(2014)). Firstly, the new $p.d.f.$ of SALD is proposed as follows.

Proposition

Let z be an ALD random variable with $z \sim ALD(\kappa, \tau, \theta)$, then the random variable $\varepsilon_t = \frac{z}{S.D.[z]}$ has scaled ALD (SALD) with probability density function given by:

$$f(\varepsilon_t|\kappa,\theta,\sigma_t) = \begin{cases} \dfrac{\sqrt{1+\kappa^4}}{1+\kappa^2}\dfrac{1}{\sigma_t}\ exp(\dfrac{-\sqrt{1+\kappa^4}}{\sigma_t}(\varepsilon_t-\theta)) & \varepsilon_t \geq \theta \\ \dfrac{\sqrt{1+\kappa^4}}{1+\kappa^2}\dfrac{1}{\sigma_t}\ exp(\dfrac{\sqrt{1+\kappa^4}}{\kappa^2\sigma_t}(\varepsilon_t-\theta)) & \varepsilon_t < \theta \end{cases} \tag{18}$$

- Then, the corresponding SMU of SALD is proposed as follows.

Proposition

If $\lambda \sim Ga(2,1)$ *and* $\varepsilon_t \sim U(\varepsilon_t|\theta - \frac{\lambda\kappa^2\sigma_t}{\sqrt{1+\kappa^4}}, \theta + \frac{\lambda\sigma_t}{\sqrt{1+\kappa^4}})$*, then the SMU density:*

$$f(\varepsilon_t|\kappa,\theta,\lambda,\sigma_t) = \int_0^\infty f_U(\varepsilon_t|\theta - \frac{\lambda\kappa^2\sigma_t}{\sqrt{1+\kappa^4}}, \theta + \frac{\lambda\sigma_t}{\sqrt{1+\kappa^4}}) \times f_{Ga}(\lambda|2,1)d\lambda \tag{19}$$

has the same form as that of SALD density function given in equation (18).

- Accordingly, the original SV model can be written hierarchically as:
  Return equation:

$$y_t|\kappa,\theta,\lambda,h_t \sim U(\theta - \frac{\lambda\kappa^2 e^{h_t/2}}{\sqrt{1+\kappa^4}}, \theta + \frac{\lambda e^{h_t/2}}{\sqrt{1+\kappa^4}}) \quad (20)$$

$$\lambda \sim Ga(2,1) \quad (21)$$

Volatility equation:

$$h_t|\alpha,\beta,\sigma_\eta^2,h_{t-1} \sim N(\alpha+\beta(h_{t-1}-\alpha),\sigma_\eta^2) \qquad t=1,2,...,T \quad (22)$$

$$h_1 \sim N(\alpha, \frac{\sigma_\eta^2}{1-\beta^2}) \quad (23)$$

Data and preliminary results

- Two major oil markets: cushing West intermediate crude oil (WTI) and Europe Brent oil (Brent).
- Daily closing spot prices are obtained from the U.S. Energy Information Administration (EIA), covering the periods from May 22, 2006 to May 20, 2016, resulting in 2520 observations in WTI and 2522 observations in Brent.
- Daily return y_t is calculated using formula: $y_t = ln(\frac{p_t}{p_{t-1}})$.

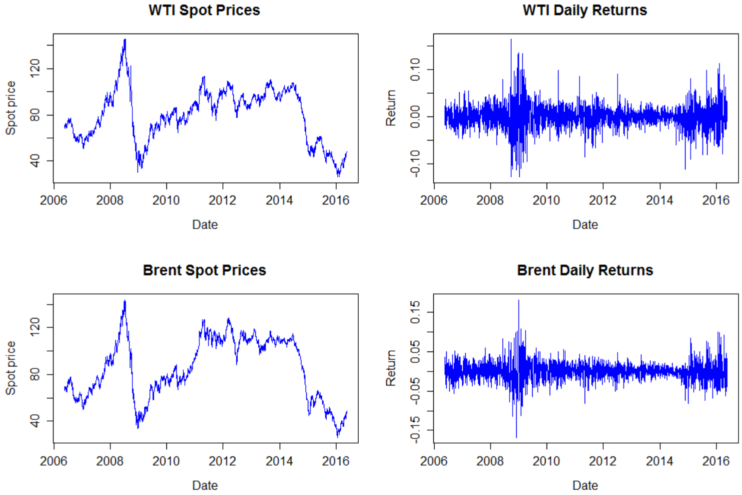

Figure 1: Daily spot prices and returns for WIT and Brent from May 2006 to May 2016

Table 1: Descriptive statistics of oil return series

	WTI	Brent
Panel A: Descriptive statistics		
Mean	-0.000144	-0.000127
Std.dev.	0.024863	0.021998
Maximum	0.164137	0.181297
Minimum	-0.128267	-0.168320
Skewness	0.1567	0.1443
Kurtosis	7.6122	8.8043
J-B test	2243.0570***	3547.5790***
Q(10)	30.6030***	16.9600*
Q(20)	60.8980***	54.2270***
ARCH(10)	475.9680***	215.7230***
ARCH(20)	575.8620***	409.0370***
Panel B: Unit roots and stationarity tests		
ADF	-51.4930***	-48.9570***
PP	-51.5220***	-48.9660***
KPSS	0.0507	0.0690

Note: Q(l) is the Ljung-Box statistic with order up to l. Test statistic of ARCH(m) is obtained using chi-squared distribution with lag up to m while ADF and PP statistics are based on t distribution. The largest value from the first 8 lags of KPSS test are listed. $*$, $**$ and $***$ denote rejection of null hypothesis at 10%, 5% and 1% significant level respectively.

Posterior estimation

- **Convergence diagnostic**: Brooks-Gelman-Rubin (BGR) diagnostic, proposed by Gelman and Rubin (1992) and developed by Brooks and Gelman (1998).

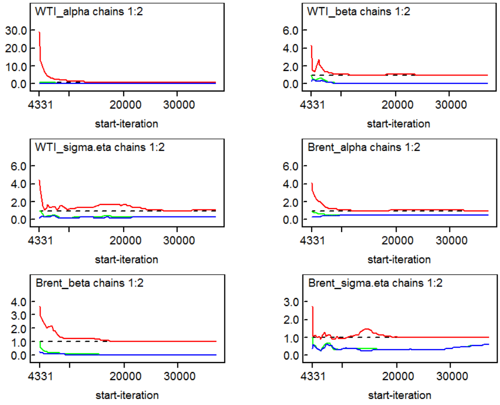

Figure 2: BGR convergence diagnostic for the parameters of SV-ALD model in WTI and Brent market

- **Posterior estimates and model comparison**:

Table 2: Posterior summary statistics of the parameters in SV-ALD and SV-N model in WTI and Brent market

Market	Parameter	Mean	SD	MC error	95% CI	DIC	Sample
				SV-ALD			
WTI	α	-7.58700	0.48730	0.00408	(-8.42200,-6.69800)	-15898.3	60000
	β	0.99470	0.00224	0.00005	(0.98990,0.99870)		
	σ_η	0.08891	0.00992	0.00051	(0.07065,0.10810)		
Brent	α	-7.75000	0.56200	0.00587	(-8.66000,-6.69600)	-16595.5	60000
	β	0.99590	0.00184	0.00004	(0.99200,0.99910)		
	σ_η	0.07351	0.00812	0.00042	(0.06106,0.09410)		
				SV-N			
WTI	α	-7.87200	0.32060	0.00218	(-8.48000,-7.26900)	-12566.3	60000
	β	0.98990	0.00378	0.00012	(0.98150,0.99640)		
	σ_η	0.12960	0.01677	0.00080	(0.10020,0.16560)		
Brent	α	-7.95400	0.49910	0.00334	(-8.76800,-7.03900)	-13028.7	60000
	β	0.99450	0.00247	0.00007	(0.98900,0.99870)		
	σ_η	0.09329	0.01252	0.00060	(0.07317,0.1207)		

- **Dynamic volatility estimates**:

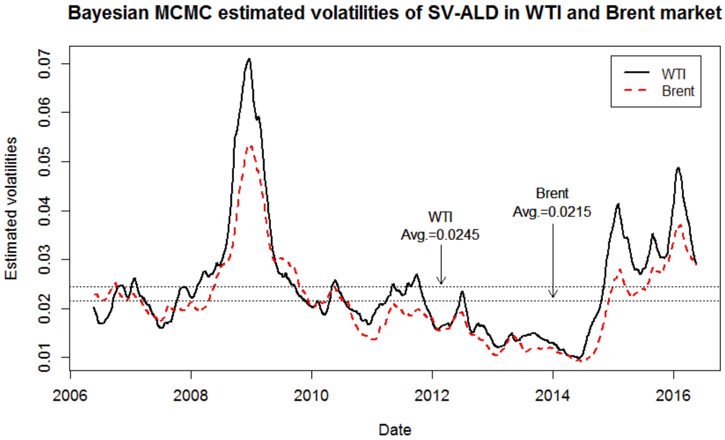

Figure 3: Bayesian MCMC estimation of the latent volatilities for WTI and Brent oil returns from May 2006 to May 2016

Dynamic VaR and CVaR estimates (Brent example)

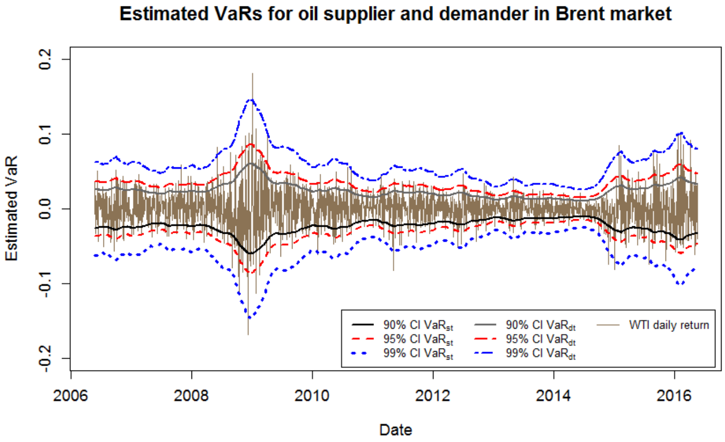

Figure 4: Dynamic VaR estimates for oil supplier and demander in Brent market using SV-ALD model at different confidence intervals

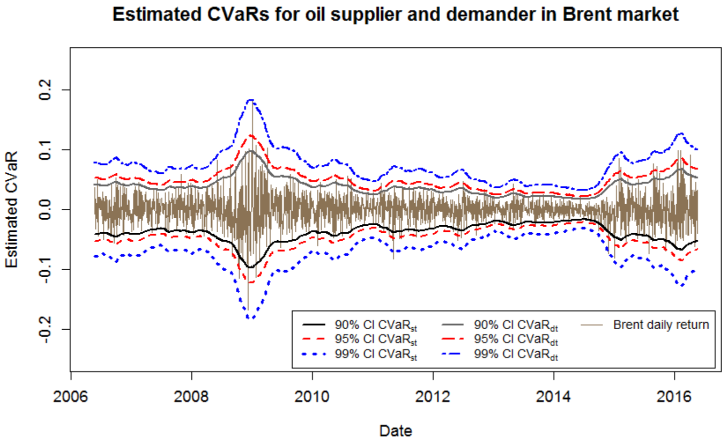

Figure 5: Dynamic CVaR estimates for oil supplier and demander in Brent market using SV-ALD model at different confidence intervals

Backtesting VaR model

- Failure rate for oil supplier and demander is defined as:

$$FRVaR_s = \frac{1}{T}\sum_{t=1}^{T} I_t(y_t < -VaR_{s,t}); \qquad FRVaR_d = \frac{1}{T}\sum_{t=1}^{T} I_t(y_t > VaR_{d,t})$$

  where $I_t(\cdot)$ is the indicator information and it equals to 1 if condition is satisfied and 0, otherwise.

- Three formal likelihood ratio statistic tests:
  - Kupiec (1995) unconditional coverage test:

$$LR_{uc} = -2\,log\left\{\alpha^N (1-\alpha)^{T-N}\right\} + 2\,log\left\{\left(\frac{N}{T}\right)^N \left(1-\frac{N}{T}\right)^{T-N}\right\}$$

  - Christoffersen (1998) independent test:

$$\begin{aligned} LR_{ind} =& 2\,log\left\{(1-\pi_{01})^{n_{00}}\pi_{01}^{n_{01}}(1-\pi_{11})^{n_{10}}\pi_{11}^{n_{11}}\right\} \\ & - 2\,log\left\{(1-\pi)^{n_{00}+n_{10}}\pi^{n_{01}+n_{11}}\right\} \end{aligned}$$

  - Christoffersen (1998) conditional coverage test:

$$LR_{cc} = -2\,log\left\{\alpha^N (1-\alpha)^{T-N}\right\} + 2\,log\left\{(1-\pi_{01})^{n_{00}}\,\pi_{01}^{n_{01}}\,(1-\pi_{11})^{n_{10}}\,\pi_{11}^{n_{11}}\right\}$$

Table 3: VaR backtesting results under SV-ALD model for WTI and Brent market at different confidence intervals

α	Risk	Failure times		Failure rate		LR_{uc}		LR_{ind}		LR_{cc}	
		WTI	Brent	WTI	Brent	WTI	Brent	WTI	Brent	WTI	Brent
10%	$VaR_{s,t}$	250	267	9.925%	10.591%	0.8995	0.3267	0.3618	0.3317	0.5896	0.3450
	$VaR_{d,t}$	252	257	10.004%	10.194%	0.9947	0.7457	0.8635	0.9635	0.8867	0.8511
5%	$VaR_{s,t}$	81	104	3.216%	4.125%	0.0000*	0.0380*	0.4058	0.8819	0.0000*	0.1101
	$VaR_{d,t}$	87	103	3.454%	4.086%	0.0002*	0.0298*	0.2738	0.1922	0.0004*	0.0387*
1%	$VaR_{s,t}$	7	8	0.279%	0.317%	0.0000*	0.0000*	0.8434	0.8214	0.0000*	0.0003*
	$VaR_{d,t}$	5	7	0.199%	0.278%	0.0000*	0.0000*	0.8878	0.8435	0.0000*	0.0000*

Note: α of 10%, 5% and 1% represent prescribed VaR level corresponding to 90%, 95% and 99% CI respectively, LR_{uc} columns show p-values of Kupiec's (1995) unconditional coverage test, LR_{ind} columns are p-values of Christofferson's (1998) independent test and LR_{cc} columns are p-values of Christofferson's (1998) conditional coverage test, * denotes significance at its corresponding risk level.

Backtesting CVaR model

- Zero-mean residual test of McNeil Frey (2000); censored Gaussian method of Berkowitz (2001) Functional delta approach of Kerkhol and Melenverg (2004).
- Backtest CVaR on the basis of equal quantiles (Kerkhol and Melenverg, 2004). Nominal risk level $\tilde{\alpha}$ that CVaR located at is needed.
  - $c.d.f$ of ALD is given by:

$$F(z|\kappa,\theta,\tau) = \begin{cases} 1 - \frac{1}{1+\kappa^2}\ exp\left(-\frac{\sqrt{2}\kappa}{\tau}(z-\theta)\right) & z \geq \theta \\ \frac{\kappa^2}{1+\kappa^2}\ exp\left(\frac{\sqrt{2}}{\tau\kappa}(z-\theta)\right) & z < \theta \end{cases} \tag{24}$$

  - The probability ($\tilde{\alpha}$) that CVaR occurs at under ALD for oil supplier and demander can be derived as:

$$\begin{aligned} Supplier: \quad \tilde{\alpha} &= F(CVaR_s|\alpha) = \frac{\alpha}{e} \\ Demander: \quad \tilde{\alpha} &= 1 - F(CVaR_d|\alpha) = \frac{\alpha}{e} \end{aligned} \tag{25}$$

Table 4: CVaR backtesting results under SV-ALD model for WTI and Brent market at different confidence intervals

$\tilde{\alpha}$	Risk	Failure times		Failure rate		LR_{uc}		LR_{ind}		LR_{cc}	
		WTI	Brent	WTI	Brent	WTI	Brent	WTI	Brent	WTI	Brent
3.68%	$CVaR_{s,t}$	54	63	2.144%	2.499%	0.0000*	0.0008*	0.1239	0.6154	0.0000*	0.0033*
	$CVaR_{d,t}$	52	61	2.064%	2.420%	0.0000*	0.0003*	0.4132	0.6747	0.0000*	0.0015*
1.84%	$CVaR_{s,t}$	23	17	0.913%	0.674%	0.0001*	0.0000*	0.5149	0.6308	0.0005*	0.0000*
	$CVaR_{d,t}$	14	19	0.556%	0.754%	0.0000*	0.0000*	0.6924	0.5911	0.0000*	0.0000*
0.37%	$CVaR_{s,t}$	3	2	0.119%	0.079%	0.0155	0.0035*	0.9326	0.9551	0.0533	0.0141
	$CVaR_{d,t}$	3	0	0.119%	0.000%	0.0155	0.0000*	0.9326	1.0000	0.0533	0.0000*

Note: $\tilde{\alpha}$ of 3.68%, 1.84% and 0.37% represent nominal CVaR level corresponding to 90%, 95% and 99% CI respectively, LR_{uc} columns show p-values of Kupiec's (1995) unconditional coverage test, LR_{ind} columns are p-values of Christofferson's (1998) independent test and LR_{cc} columns are p-values of Christofferson's (1998) conditional coverage test, * denotes significance at its corresponding risk level.

Concluding remarks

- The AL distribution (Kotz et al., 2001) is extended as an error distribution to the return equation of discrete SV model, based on which the analytic expressions of VaR and CVaR for oil supplier and demander are derived to measure oil market risks.
- In Bayesian MCMC, the proposition of SMU representation of ALD facilitates an efficient Gibbs sampling algorithm, making Bayesian statistical inferences easy to implement.
- Heavy-tailed SV-ALD model is more capable to fit oil return series than SV-N model.
- Practicability of CVaR backtesting in SV-ALD framework by working on the basis of equal quantiles, $\tilde{\alpha}$ is proved depending only on α.
- VaR and CVaR demonstrate a conservative manner in measuring maket risk of oil rices under SV-ALD when focuing on extreme rail risks (Also see Chen et al., 2012).

Thank you for your attention !