Systemic Risk and Interbank Lending

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Outline

Coupled Feller Diffusions: Systemic Risk

$X_t^{(i)}, i = 1, \ldots, N$ denote monetary reserves of N banks

$$dX_t^i = \left(\frac{a}{N} \sum_{j=1}^{N} (X_t^j - X_t^i) + \gamma_t \right) dt + 2\sqrt{X_t^i} dW_t^i \,, \quad i = 1, \cdots, N.$$

Assume **independent Brownian motions** $W_t^{(i)}, i = 1, \ldots, N$ and **identical constant volatilities** $\sigma_t^{(i)} = \sigma$

- The nonnegative **growth rate** γ_t is a deterministic function in L_∞ space.
- The overall **rate of borrowing and lending** a/N has been normalized by the number of banks and we assume $a > 0$
- Denote the **default level** by $D = 0$.

The Case of Constant Growth Rate

Assume bank i has the monetary reserve X_t^i with a constant growth rate γ written as

$$dX_t^i = \left(\frac{a}{N} \sum_{j=1}^{N} (X_t^j - X_t^i) + \gamma \right) dt + 2\sqrt{X_t^i} dW_t^i \,, \quad i = 1, \cdots, N, \tag{1}$$

where the lending preference is a fixed normalized constant $\frac{a}{N} \leq 1$ seen as one particular case in Fouque-Ichiba (2012). Consequently, the total monetary reserve $Y_t = \sum_{i=1}^{N} X_t^i$ is given by

$$dY_t = N\gamma dt + 2\sqrt{Y_t} d\widetilde{W}_t$$

where $\widetilde{W}_t$ is a standard Brownian motion in some extension probability space by Lévy theorem.

Financial Implication

- If $\gamma > \frac{2}{N}$, systemic risk never happens owing to the total monetary reserve Y never reaching zero:

$$\boldsymbol{P}(Y_t > 0 \text{ for all } t \in [0, \infty)) = 1. \tag{2}$$

- If $\gamma = \frac{2}{N}$, we obtain

$$\boldsymbol{P}(\limsup_{t \to \infty} Y_t = \infty) = 1, \tag{3}$$

so that the system survivals since the total reserve X stays in positive . However, due to

$$\boldsymbol{P}(\inf_{0 \leq t \leq \infty} Y_t = 0) = 1, \tag{4}$$

the shrinking of the total reserve to almost nothing leads to a financial crisis at some point in the future almost surely.

Financial Implication

- If $0 < \gamma < \frac{2}{N}$, the total monetary reserve Y does not satisfy the property (3) but qualify (4). All banks face defaults in the future since the total monetary reserve Y attains zero at some large time almost surely and reflects instantaneously at the point $\{0\}$.
- If $\gamma = 0$, the total monetary reserve Y approaches to zero in the finite time leading to all banks default at some time and stay zero thereafter almost surely.

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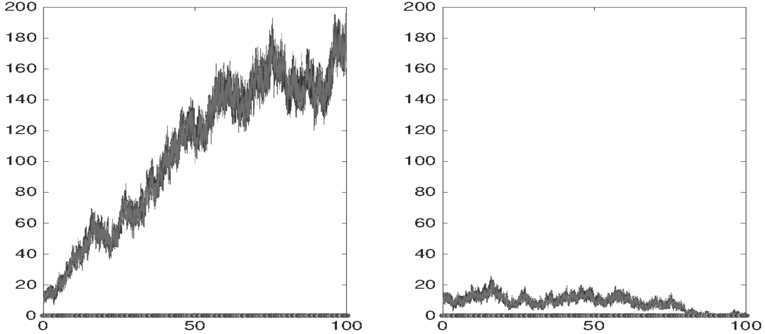
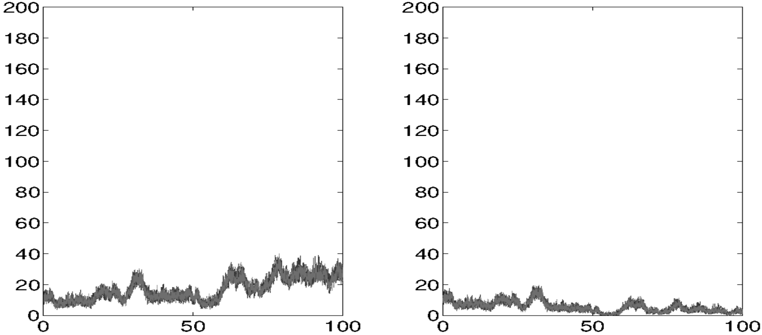


One realization of $N = 10$ trajectories of (1) with the common mean-reverting rate $a = 10$ and with$\gamma = 2$ (left plot) and with $\gamma = 0$ (right plot).



One realization of $N = 10$ trajectories of (1) with the common mean-reverting rate $a = 10$ and with$\gamma = 0.2$ (left plot) and with $\gamma = 0$ (right plot).

Stability

Note that given the identical growth rate γ and the symmetric lending preference a/N, the matrix-valued process $(X_t^i - X_t^j)_{1\leq i,j\leq N}$ is stochastically stable using Proposition 2.4 and Corollary 2.1 in Fouque-Ichiba (2012).

Financial Implication

The monetary reserve X_t^i can be rewritten as

$$
\begin{aligned}
dX_t^i &= \left(\frac{a}{N} \sum_{j=1}^{N} (X_t^j - X_t^i) + \gamma \right) dt + 2\sqrt{X_t^i} dW_t^i \\
&= \left(a(\overline{X}_t - X_t^i) + \gamma \right) + 2\sqrt{X_t^i} dW_t^i , \quad i = 1, \cdots, N,
\end{aligned}
$$

where X_t^i is mean-reverting at the averaged capitalization $\frac{1}{N} Y_t$.

"In the case of $\gamma > \frac{2}{N}$, the behavior of lending and borrowing creates stability; however, in the case of $0 \leq \gamma \leq \frac{2}{N}$, this type of interacting leads to systemic risk."

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Stochastic Games

If the banks are allowed to have their own strategies in the system, such as minimizing costs, maximizing profits,...

- Can we find an equilibrium in which the previous analysis can still be performed?
- Can we find and characterize a Nash equilibrium?

Types of admissible controls

Admissible controls can be identified as four classes:

- The admissible controls are Open Loop if the control α_t is a function of t and X_0.
- The admissible controls are Closed Loop Perfect State if the control α_t is a function of t and state history $X_{[0,t]}$.
- The admissible controls are Momeryless Perfect State if the control α_t is a function of t, X_0 and X_t.
- The admissible controls are Feedback Perfect State or Markovian if the control α_t is a function of t and X_t.

See also LECTURES ON BSDES, STOCHASTIC CONTROL, AND STOCHASTIC DIFFERENTIAL GAMES WITH FINANCIAL APPLICATIONS by R. Carmona

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Stochastic Games

Denoting $\overline{X}_t = \frac{1}{N}\sum_i^N X_t^i$, the dynamics is

$$dX_t^i = a(\overline{X}_t - X_t^i)dt + \alpha_t^i dt + 2\sqrt{X_t^i}dW_t^i, \ i = 1, \cdots, N$$

where α^i is the control of bank i, the rate of lending to and borrowing from a central bank given by **minimizing**

$$J^i(\alpha^1, \cdots, \alpha^N) = \boldsymbol{E}\left\{\int_0^T f_i(X_t, \alpha_t^i)dt + g_i(X_T^i)\right\},$$

with **running cost**

$$f_i(x, \alpha^i) = \left[\frac{1}{2}(\alpha^i)^2 - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2\right], \quad q^2 \leq \epsilon,$$

and **terminal cost** $\qquad g_i(x) = \frac{c}{2}\left(\overline{x} - x^i\right)^2.$

Markov Nash Equilibria (HJB/PDE Approach)

In the **Markovian** setting, given the optimal strategies $\hat{\alpha}^j$ for $j \neq i$ with the corresponding trajectories

$$\hat{X}^{-i} = (\hat{X}^i, \cdots, \hat{X}^{i-1}, \hat{X}^{i+1}, \cdots, \hat{X}^N),$$

the value function of bank i is written as

$$V^i(t,x) = \inf_{\alpha^i \in \mathcal{A}^i} \boldsymbol{E}_{t,x} \left\{ \int_t^T f_i(X_s, \hat{X}_s^{-i}, \alpha_s^i) ds + g_i(X_T^i, \hat{X}_T^{-i}) \right\},$$

with **the running cost** functions f_i

$$f_i(x, \alpha^i) = \left[\frac{1}{2}(\alpha^i)^2 - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2 \right], \quad q^2 \leq \epsilon,$$

and **the terminal cost** $g_i(x) = \frac{c}{2}\left(\overline{x} - x^i\right)^2$. Note that $\alpha_\cdot$ is a progressively measurable control process and $\alpha_\cdot^i$ is admissible if for $0 \leq t \leq T$,

$$X_t^{j,\alpha^i} \geq 0, \ \forall j \neq i \text{ and } X_t^{i,\alpha^i} \geq 0 \ a.s..$$

The HJB Equation

Using the dynamic programming principle in search for a equilibrium, given the optimal control $\hat{\alpha}^j$ for $j \neq i$, the corresponding **HJB equation** reads

$$
\begin{aligned}
\partial_t V^i \quad &+ \quad \inf_{\alpha^i} \Bigg\{ \sum_{j \neq i} \left[a\left(\overline{x} - x^j\right) + \hat{\alpha}^j(t,x) \right] \partial_{x^j} V^i \\
&+ \quad \left[a\left(\overline{x} - x^i\right) + \alpha^i \right] \partial_{x^i} V^i \\
&+ \quad 2 \sum_{j=1}^{N} x^j \partial_{x^j x^j} V^i \\
&+ \quad \frac{(\alpha^i)^2}{2} - q\alpha^i \left(\overline{x} - x^i\right) + \frac{\epsilon}{2}(\overline{x} - x^i)^2 \Bigg\} = 0,
\end{aligned}
$$

with terminal conditions $V^i(T,x) = \frac{c}{2}(\overline{x} - x^i)^2$.

The HJB Equation

The first order condition gives the **optimal strategies** $\hat{\alpha}^i = q(\overline{x} - x^i) - \partial_{x^i} V^i$ for banks $i = 1, \cdots, N$, so that the HJB equations V^i for $i = 1, \cdots, N$ become

$$
\begin{aligned}
\partial_t V^i \quad &+ \quad \sum_{j=1}^{N} \left[(a+q)\left(\overline{x} - x^j\right) - \partial_{x^j} V^j \right] \partial_{x^j} V^i \\
&+ \quad 2 \sum_{j=1}^{N} x^j \partial_{x^j x^j} V^i \\
&+ \quad \frac{1}{2}(\epsilon - q^2)\left(\overline{x} - x^i\right)^2 + \frac{1}{2}(\partial_{x^i} V^i)^2 = 0.
\end{aligned}
$$

Ansatz

$$
V^i(t,x) = \frac{\eta_t^c}{2}(\overline{x} - x^i)^2 + L_t^c(\overline{x} - x^i) + \phi_t^c \overline{x} + \mu_t^c,
$$

where η_t^c, L_t^c, ϕ_t^c and μ_t^c are deterministic functions.

Solution to HJB/PDE Approach

Plugging into the HJB equation, η_t^c, L_t^c, ϕ_t^c and μ_t^c satisfy the system of ordinary differential equations

$$
\begin{aligned}
\dot{\eta}_t^c &= 2(a+q)\eta_t^c + (1-\frac{1}{N^2})(\eta_t^c)^2 - (\epsilon - q^2),\\
\dot{L}_t^c &= \left(a+q+\frac{1}{N}(1-\frac{1}{N})\eta_t^c\right) L_t^c + \frac{1}{N}(1-\frac{1}{N})\eta_t^c\phi_t^c + 2(1-\frac{2}{N})\eta_t^c,\\
\dot{\phi}_t^c &= -2(1-\frac{1}{N})\eta_t^c,\\
\dot{\mu}_t^c &= \frac{1}{N}(1-\frac{1}{2N})(\phi_t^c)^2 - \frac{1}{2}(1-\frac{1}{N})^2(L_t^c)^2 - (1-\frac{1}{N})^2 L_t^c\phi_t^c - \gamma_t\phi_t^c,
\end{aligned}
$$

with terminal conditions $\eta_T^c = c$, $L_T^c = 0$, $\phi_T^c = 0$, and $\mu_T^c = 0$.

Solution to the Riccati Equation

$$\eta_t^c = \frac{-(\epsilon - q^2)\left(e^{(\delta^+-\delta^-)(T-t)} - 1\right) - c\left(\delta^+ e^{(\delta^+-\delta^-)(T-t)} - \delta^-\right)}{\left(\delta^- e^{(\delta^+-\delta^-)(T-t)} - \delta^+\right) - c(1 - \frac{1}{N^2})\left(e^{(\delta^+-\delta^-)(T-t)} - 1\right)},$$

with the notations

$$\begin{aligned} \delta^{\pm} &= -(a+q) \pm \sqrt{R}, \\ R &= (a+q)^2 + \left(1 - \frac{1}{N^2}\right)(\epsilon - q^2) > 0. \end{aligned}$$

Observe that $\tilde{\eta}_t$ is well defined for any $t \leq T$ since the denominator can be written as

$$-\left(e^{(\delta^+-\delta^-)(T-t)} + 1\right)\sqrt{R} - \left(a + q + c\left(1 - \frac{1}{N^2}\right)\right)\left(e^{(\delta^+-\delta^-)(T-t)} - 1\right),$$

which stays negative because $\delta^+ - \delta^- = 2\sqrt{R} > 0$.
In fact, using $q^2 \leq \epsilon$, we see that η_t^c is positive with $\eta_T^c = c$.

Markov Nash Equilibrium

Once the function η_t^c has been obtained, bank i implements its strategy by using its control

$$\hat{\alpha}_t^i = \left(q + (1 - \frac{1}{N})\eta_t^c\right)(\overline{X}_t - X_t^i) - \psi_t^c,$$

where $\psi_t^c = (\frac{1}{N} - 1)L_t^c + \frac{1}{N}\phi_t^c \geq 0$ is treated as the deposit rate. we obtain

$$dX_t^i = \left\{ \left(a + q + (1 - \frac{1}{N})\eta_t^c\right)(\overline{X}_t - X_t^i) + \gamma_t - \psi_t^c \right\} dt + 2\sqrt{X_t^i}dW_t^i,$$

and the total monetary reserve $Y_t = \sum_{i=1}^N X_t^i$ written as

$$dY_t = N(\gamma_t - \psi_t^c)dt + 2\sqrt{Y_t}d\widetilde{W}_t,$$

with a standard Brownian motion $\widetilde{W}_t$ in some extension probability space.

Regularity and Admissibility

In order to satisfy the regularity condition of X_t^i and guarantee the admissibility of $\hat{\alpha}^i$ for $i = 1, \cdots, N$, we further assume

$$\gamma_t \geq \psi_t^c,$$

and

$$a + q + (1 - \frac{1}{N})\eta_t^c \leq N,$$

for $0 \leq t \leq T$.

Verification Theorem

Given the optimal strategies $\hat{\alpha}^j$ given by

$$\hat{\alpha}^j_t = \left(q + (1 - \frac{1}{N})\eta^c_t \right) (\overline{X}_t - X^j_t) - \psi^c_t, \tag{5}$$

for all $j \neq i$, V^i given by

$$V^i(t,x) = \frac{\eta^c_t}{2}(\overline{x} - x^i)^2 + L^c_t(\overline{x} - x^i) + \phi^c_t \overline{x} + \mu^c_t,$$

is the value function associated to the problem

$$V^i(t,x) = \inf_{\alpha^i \in \mathcal{A}^i} \boldsymbol{E}_{t,x} \left\{ \int_t^T f_i(X_s, \hat{X}^{-i}_s, \alpha^i_s) ds + g_i(X^i_T, \hat{X}^{-i}_T) \right\},$$

subject to

$$dX^i_t = a(\overline{X}_t - X^i_t)dt + \alpha^i_t dt + 2\sqrt{X^i_t} dW^i_t, \ \ i = 1, \cdots, N$$

and $\hat{\alpha}^i$ in the form of (5) is the optimal strategy for bank i and also the Markov Nash equilibrium.

Financial Implications

Under the instability condition

$$\sup_{0\leq t\leq T}(\gamma_t - \psi_t^c) < \frac{2}{N},$$

denoting the first passage time $\tau = \inf\{t > 0; Y_t = 0\}$, we discuss the tail probability of all defaults based on the total monetary reserve Y_t and its corresponding first passage time

$$\tau = \inf\{t > 0; Y_t = 0\}.$$

Financial Implications

Applying the comparison theorem studied in, Ikeda-Watanabe(1977), Li-Yor(1999), and Yor(2003), we estimate the tail probability of all defaults written as

$$\begin{aligned}\Gamma\left(\frac{Y_0^2}{2T}; N\inf_{0\le t\le T}(\gamma_t-\psi_t^c)\right) &= \boldsymbol{P}(\check{\tau}>T)\\ &\le \boldsymbol{P}(\tau>T)\\ &\le \boldsymbol{P}(\hat{\tau}>T)=\Gamma\left(\frac{Y_0^2}{2T}; N\sup_{0\le t\le T}(\gamma_t-\psi_t^c)\right),\end{aligned}$$

for $T\ge t$ where $\Gamma(s,x):=\int_0^x u^{s-1}e^{-u}du$ for $s>0$, and $x\ge 0$ where $\hat{\tau}=\inf\{t>0;\hat{Y}_t=0\}$ and $\check{\tau}=\inf\{t>0;\check{Y}_t=0\}$ with corresponding dynamics

$$d\hat{Y}_t=N\sup_{0\le t\le T}(\gamma_t-\psi_t^c)dt+2\sqrt{\hat{Y}_t}d\widetilde{W}_t,$$

and

$$d\check{Y}_t=N\inf_{0\le t\le T}(\gamma_t-\psi_t^c)dt+2\sqrt{\check{Y}_t}d\widetilde{W}_t.$$

Financial Implications

- The optimal strategy yields the increasing liquidity qualified by the larger rate of lending and borrowing given by

$$A_t = a + q + (1 - \frac{1}{N})\eta_t^c.$$

- Due to the volatility driven by the square root function of states written as $\sqrt{X_t^i}$, according to the solution of the optimal strategy, bank i intends to borrow less money from or to lend more money to a central bank to manipulate its own reserve through the positive deposit rate

$$dX_t^i = \left\{ \left(a + q + (1 - \frac{1}{N})\eta_t^c \right) (\overline{X}_t - X_t^i) + \gamma_t - \psi_t^c \right\} dt + 2\sqrt{X_t^i} dW_t^i$$

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so that a central bank acts as a central deposit corporation.

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- Notice that if T is large, η_t^c is treated as a constant and ψ_t^c is almost linear in t.

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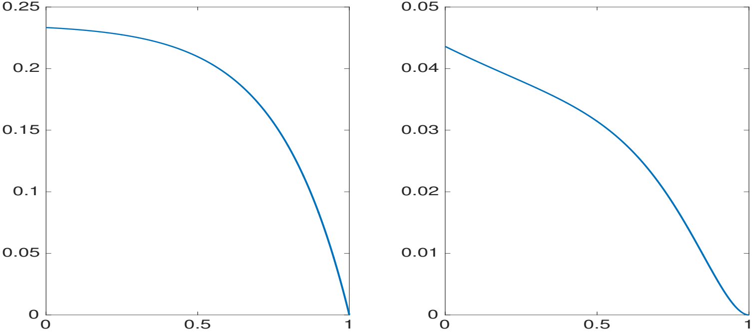
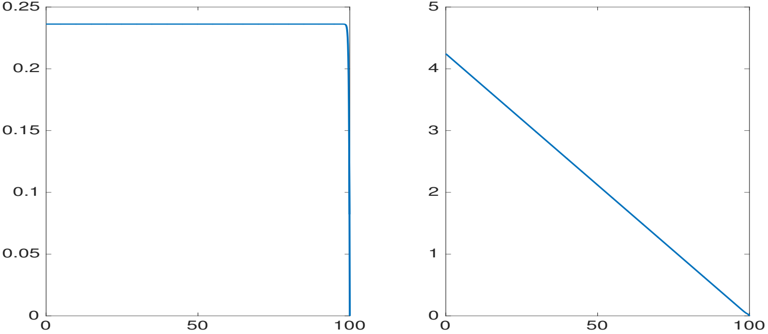


Plots of η_t on the left and ψ_t on the right for $0 \leq t \leq T$ with $a = 1$, $q = 1$, $\epsilon = 2$, $c = 0$, $N = 10$, $T = 1$.



Plots of η_t on the left and ψ_t on the right for $0 \leq t \leq T$ with $a = 1$, $q = 1$, $\epsilon = 2$, $c = 0$, $N = 10$, $T = 100$.

Financial Implications

- Observe that the equilibrium creates the possibility of all defaults through ψ^c since all banks behave more conservative.
- Under the stability condition

$$\inf_{0\le t\le T}(\gamma_t - \psi^c_t) > \frac{2}{N}$$

the increasing liquidity leads to stability.

- However, if the system stays in the worse case given by

$$\sup_{0\le t\le T}(\gamma_t - \psi^c_t) \le \frac{2}{N}$$

the system of interbank lending and borrowing may face systemic risk immediately due to the large liquidity.

Financial Implications

Propsition

In the case of $\gamma_t = \gamma$, $N \to \infty$ and $a = c = 0$, if for fixed ϵ, the incentive q satisfies

$$\frac{\sqrt{2\epsilon}}{\sqrt{\gamma + 2}} < q \leq \sqrt{\epsilon},$$

or for fixed q, the running penalty ϵ satisfies

$$q^2 \leq \epsilon < \frac{1}{2}(\gamma + 2)q^2,$$

the stability condition

$$\inf_{0 \leq t \leq T} (\gamma_t - \psi_t^c) > \frac{2}{N},$$

holds.

ϵ-Nash Equilibria: Mean Field Games

In MFGs, we solve the ϵ-Nash equilibria in the mean field limit $N \to \infty$. According to the steps in Carmona-Fouque-Sun(2013):

1. fix $(m_t)_{t\geq 0}$ as a candidate for the limit $\overline{X}_t$ as $N \to \infty$

$$m_t = \lim_{N\to\infty} \overline{X}_t.$$

2. Solve the one-player standard control problem

$$V^m(t,x) = \inf_{\alpha_t} \boldsymbol{E}_{t,x}\left\{ \int_0^T \left[\frac{\alpha_t^2}{2} - q\alpha_t(m_t - X_t) + \frac{\epsilon}{2}(m_t - X_t)^2\right] dt + \frac{c}{2}(m_T - X_T)^2 \right\},$$

subject to

$$dX_t = \Big(a(m_t - X_t) + \gamma_t\Big)dt + \alpha_t dt + 2\sqrt{X_t}dW_t,$$

with the nonnegative deterministic growth rate γ_t in L_∞ space where W_t is a standard Brownian motion independent of the initial value $X_0 = \xi_0$ which may be a square integrable random variable.

3. Finally, we obtain m_t such that $m_t = \boldsymbol{E}[X_t]$ for all $0 \leq t \leq T$.

ϵ-Nash Equilibria: Mean Field Games

Given the candidate m_t, the corresponding HJB equation is written as

$$\partial_t V^m + \inf_{\alpha} \Big\{ \left(a(m_t - x) + \gamma_t + \alpha\right) \partial_x V^m + 2x\partial_{xx} V^m + \frac{\alpha^2}{2} - q\alpha(m_t - x) + \frac{\epsilon}{2}(m_t - x)^2 \Big\} = 0.$$

The first order condition leads to optimal strategy given by

$$\hat{\alpha}_t = q(m_t - X_t) - \partial_x V^m.$$

Plugging $\hat{\alpha}_t$ into the above HJB gives

$$\partial_t V^m + \left((a+q)(m_t - x) - \partial_x V^m + \gamma_t\right) \partial_x V^m + 2x\partial_{xx} V^m + \frac{(\partial_x V^m)^2}{2} + \frac{\epsilon - q^2}{2}(m_t - x)^2 = 0.$$

ϵ-Nash Equilibria: Mean Field Games

The ansatz

$$V^m(t,x) = \eta_t^m(m_t - x) + L_t^m(m_t - x) + \phi_t^m m_t + \mu_t^m,$$

where η_t^m, L_t^m, and μ_t^m must satisfy

$$\begin{aligned}
\dot{\eta}_t^m &= 2(a+q)\eta_t^m + (\eta_t^m)^2 - (\epsilon - q^2), \\
\dot{L}_t^m &= (a+q)L_t^m + 2\eta_t^m, \\
\dot{\phi}_t^m &= -2\eta_t^m, \\
\dot{\mu}_t^m &= -\frac{1}{2}(L_t^m)^2 - L_t^m\phi_t^m - \gamma_t\phi_t^m,
\end{aligned}$$

with terminal conditions $\eta_T^m = c$, $L_T^m = 0$, $\phi_T^m = 0$, and $\mu_T^m = 0$ leading to the ϵ-Nash equilibrium written as

$$\hat{\alpha}_t = (q + \eta_t^m)(m_t - X_t) - \psi_t,$$

with deposit rate $\psi_t = -L_t^m \geq 0$ where we assume $\gamma_t - \psi_t \geq 0$ for all $0 \leq t \leq T$.

ϵ-Nash Equilibria: Mean Field Games

- The ϵ-Nash equilibrium (6) is the exact limit of the Markov equilibrium in the finite player games as $N \to \infty$.
- We naturally regard the ϵ-Nash equilibrium as an asymptotic solution of the optimal strategy for bank i written as

$$\hat{\alpha}_t^i = (q + \eta_t^m)(\overline{X}_t - X_t^i) - \psi_t,$$

satisfying $a + q + \eta_t \leq N$ for all $0 \leq t \leq T$ in order to guarantee admissibility.

Further Discussions-I

- Compared to the admissibility conditions given by

$$X_t^{j,\alpha^i} \geq 0, \ \forall j \neq i \text{ and } X_t^{i,\alpha^i} \geq 0 \ a.s.,$$

  an individual bank may consider the more restricted admissible set written as

$$X_t^{j,\alpha^i} \geq 0, \ \forall j \neq i \text{ and } X_t^{i,\alpha^i} > 0 \ a.s..$$

  Then, the sufficient conditios for $X_t^i > 0$ for $0 \leq t \leq T$ must be rewritten.

- Referring to Espinosa and Touzi (2015), we can study the strategy under the relative performance given by

$$(1-\lambda)X_t^i + \lambda(X_t^i - \overline{X}_t) = X_t^i - \lambda\overline{X}_t.$$

Further Discussions-II

- The dynamics of monetary reserves for bank i is

$$dX_t^i = \left(\alpha_t^i - \alpha_{t-\tau}^i + \gamma_t\right) dt + 2\sqrt{X_t^i} dW_t^i\,, \quad i = 1, \cdots, N,$$

where $\alpha_{t-\tau}^i$ is the debt obligations in the form of the delayed control with $\alpha_{t-\tau}^i = 0$ for $0 \leq t < \tau$.

- Based on the results in Carmona-Fouque-Moustavi-Sun(2017), given X_t and α_θ, $\theta \in [t-\tau, t)$, the closed-loop Nash equilibrium is given by the value function written as

$$V^i(t,x) = \inf_{\alpha} \mathbf{E}_{t,x}\left\{\int_t^T f_i(X_s, \alpha_s^i)ds \mid X_t, \alpha_\theta, \theta \in [t-\tau, t)\right\},$$

where

$$f_i(X_t, \alpha_t^i) = \frac{(\alpha_t^i)^2}{2} - q\alpha_t^i(\overline{X}_t - X_t^i) + \frac{\epsilon}{2}(\overline{X}_t - X_t^i)^2,$$

where $q^2 \leq \epsilon$ so that $f_i(x,\alpha)$ is convex in (x,α) and $\alpha_.$ is a progressively measurable control process and $\alpha_.^i$ is admissible if for $0 \leq t \leq T$,

$$X_t^{j,\alpha^i} \geq 0,\ \forall j \neq i \text{ and } X_t^{i,\alpha^i} \geq 0\ a.s..$$

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THANKS FOR YOUR ATTENTION