Parisian reflection and applications in insurance and credit risk [1]

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[1] joint work with F. Avram and J. L. Pérez

Modeling ruin/default

- Classical ruin:

$$\inf\{t > 0 : X(t) < 0\}$$

  happens as soon as the risk process (or asset value) goes below a certain level.

- Parisian ruin:

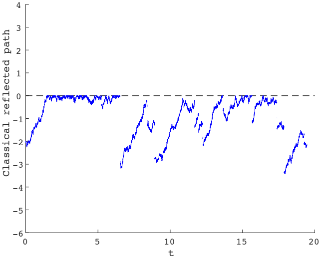
$$\inf\{t > 0 : \sup_{u \in [t-\delta, t]} X(u) < 0\}$$

  happens as soon as it stays below a certain level consecutively for δ units of time.

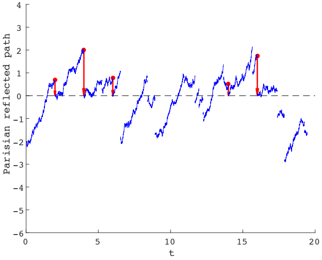
Modeling capital injection/bail-out

- Classical reflection (capital injection)
  - assume that we can observe the process continuously;
  - $L(t)$ so that $X(t) + L(t)$ is a reflected process with barrier 0;
  - minimum amount required to push upward so that it stays above 0.
- Parisian reflection (capital injection):
  - assume that we can observe the process only at intervals;
  - push the process up to 0 whenever the process is below 0 at the discrete observation times.
  - If the interval is exponentially distributed, reflection happens when the process stays below 0 consecutively for an exponential time.

Sample paths



classical reflection



Parisian reflection

Our settings

- Let $a < 0$ (can be changed to any values).
- Absolute ruin := first time the risk process (asset value etc) goes below a.
- Shareholder tries to avoid absolute ruin.
- Capital injection can be made only at independent Poisson arrival times with rate $r > 0$.
- Parisian reflection strategy:
  - pushes it up to 0 at the Poisson arrival times at which it is below zero.
  - absolute ruin can still occur (its chance vanishes as $r \uparrow \infty$).

Lévy processes $X_r(t)$ with Parisian reflection below

- $\mathcal{T}_r = \{T(i); i \geq 1\}$: jump times of an independent Poisson process with rate $r > 0$.
- We have

$$X_r(t) = X(t), \quad 0 \leq t < T_0^-(1)$$

where

$$T_0^-(1) := \inf\{T(i) : X(T(i)) < 0\}.$$

- The process then jumps upward by $|X(T_0^-(1))|$ s.t. $X_r(T_0^-(1)) = 0$.
- For $T_0^-(1) \leq t < T_0^-(2) := \inf\{T(i) > T_0^-(1) : X_r(T(i)-) < 0\}$,

$$X_r(t) = X(t) + |X(T_0^-(1))|.$$

Decompositions

- Suppose $R_r(t)$ is the cumulative amount of (Parisian) reflection until time $t \geq 0$.
- Then we have

$$X_r(t) = X(t) + R_r(t), \quad t \geq 0,$$

with

$$R_r(t) := \sum_{T_0^-(i) \leq t} |X_r(T_0^-(i)-)|, \quad t \geq 0.$$

With additional classical reflection above

- $b > 0$: dividend boundary, above which company spends all the money.
- Define classical reflected process:

$$Y^b(t) := X(t) - L^b(t)$$

where

$$L^b(t) := \sup_{0 \leq s \leq t} (X(s) - b) \vee 0, \quad t \geq 0.$$

- We have

$$Y_r^b(t) = Y^b(t), \quad 0 \leq t < \widehat{T}_0^-(1)$$

where $\widehat{T}_0^-(1) := \inf\{T(i) : \ Y^b(T(i)) < 0\}$.

- The process then jumps upward by $|Y^b(\widehat{T}_0^-(1))|$ s.t. $Y_r^b(\widehat{T}_0^-(1)) = 0$.
- For $\widehat{T}_0^-(1) \leq t < \widehat{T}_0^-(2) := \inf\{T(i) > \widehat{T}_0^-(1) : \ Y_r^b(T(i)-) < 0\}$, $Y_r^b(t)$ is the reflected process of $X(t) - X(\widehat{T}_0^-(1))$.

Decompositions

- $R_r^b(t)$: the cumulative amounts of Parisian reflection from below until time t.
- $L_r^b(t)$: the cumulative amounts of classical reflection from above until time t.
- It is clear that it admits a decomposition

$$Y_r^b(t) = X(t) + R_r^b(t) - L_r^b(t), \quad t \geq 0.$$

What we want to compute for X_r

- Define absolute ruin time:

$$\tau_a^-(r) := \inf\{t > 0 : X_r(t) < a\}, \quad a < 0,$$

  and the up-crossing time:

$$\tau_b^+(r) := \inf\{t > 0 : X_r(t) > b\}, \quad b > 0.$$

- Joint Laplace transform of $\tau_a^-(r)$, $\tau_b^+(r)$, and R_r:

$$\mathbb{E}_x\left(e^{-q\tau_b^+(r)-\theta R_r(\tau_b^+(r))}; \tau_b^+(r) < \tau_a^-(r)\right),$$

$$\mathbb{E}_x\left(e^{-q\tau_a^-(r)-\theta R_r(\tau_a^-(r))}; \tau_a^-(r) < \tau_b^+(r)\right).$$

- Expected discounted value of capital injections:

$$\mathbb{E}_x\left(\int_0^{\tau_b^+(r)\wedge\tau_a^-(r)} e^{-qt} dR_r(t)\right).$$

- And their limits as $a \downarrow -\infty$ and $b \uparrow \infty$.

What we want to compute for Y_r^b

- Define absolute ruin time:

$$\eta_a^-(r) := \inf\{t > 0 : Y_r^b(t) < a\}, \quad a < 0.$$

- Joint Laplace transform of $\eta_a^-(r)$ and R_r^b:

$$\mathbb{E}_x \left(e^{-q\eta_a^-(r) - \theta R_r^b(\eta_a^-(r))}; \eta_a^-(r) < \infty \right).$$

- Expected discounted value of dividends:

$$\mathbb{E}_x \left(\int_0^{\eta_a^-(r)} e^{-qt} dL_r^b(t) \right).$$

- Expected discounted value of capital injections:

$$\mathbb{E}_x \left(\int_0^{\eta_a^-(r)} e^{-qt} dR_r^b(t) \right).$$

- And their limits as $a \downarrow -\infty$.

Spectrally Negative Lévy Processes

- Let X be a spectrally negative Lévy process with a Laplace exponent:

$$\begin{aligned}\kappa(s) &:= \log \mathbb{E}\left[e^{sX(1)}\right] \\ &= cs + \frac{1}{2}\sigma^2 s^2 + \int_{(-\infty,0)} (e^{sz} - 1 - sz1_{\{-1<z<0\}})\nu(\mathrm{d}z),\end{aligned}$$

  such that $\int_{(-\infty,0)}(1 \wedge z^2)\nu(\mathrm{d}z) < \infty$.
- It has paths of bounded variation if and only if $\sigma = 0$ and $\int_{(-1,0)} z\,\nu(\mathrm{d}z) < \infty$.
- We exclude the case X is a subordinator.

Scale Functions

- Recall that X is a spectrally negative Lévy process with Laplace exponent $\kappa(s) = \log \mathbb{E}\left[e^{sX(1)}\right]$.
- Fix any $q \geq 0$, there exists a function called the q-scale function

  $$W_q : \mathbb{R} \to [0, \infty),$$

  which is zero on $(-\infty, 0)$, continuous and strictly increasing on $[0, \infty)$, and is characterized by the Laplace transform:

  $$\int_0^\infty e^{-sx} W_q(x) \mathrm{d}x = \frac{1}{\kappa(s) - q}, \qquad s > \Phi_q,$$

  where

  $$\Phi_q := \sup\{\lambda \geq 0 : \kappa(\lambda) = q\}.$$

Scale Functions (Cont'd)

Let us define the first down- and up-crossing times, respectively, by

$$\tau_a^- := \inf\{t \geq 0 : X(t) < a\}$$
$$\tau_b^+ := \inf\{t \geq 0 : X(t) > b\}.$$

Then we have for any $b > 0$

$$\mathbb{E}_x\left[e^{-q\tau_b^+} 1_{\{\tau_b^+ < \tau_0^-\}}\right] = \frac{W_q(x)}{W_q(b)},$$
$$\mathbb{E}_x\left[e^{-q\tau_0^-} 1_{\{\tau_b^+ > \tau_0^-\}}\right] = Z_q(x) - Z_q(b)\frac{W_q(x)}{W_q(b)},$$

where

$$\overline{W}_q(x) := \int_0^x W_q(y)\mathrm{d}y,$$
$$Z_q(x) := 1 + q\overline{W}_q(x).$$

Scale Functions (Cont'd)

- We also define, for $x \in \mathbb{R}$,

$$\overline{Z}_q(x) := \int_0^x Z_q(z) dz.$$

- Define also, for all $x \in \mathbb{R}$ and $a \leq 0$,

$$W^a_{\alpha,\beta}(x) := W_{\alpha+\beta}(x-a) - \beta \int_0^x W_\alpha(x-y) W_{\alpha+\beta}(y-a) \mathrm{d}y,$$

$$Z^a_{\alpha,\beta}(x) := Z_{\alpha+\beta}(x-a) - \beta \int_0^x W_\alpha(x-y) Z_{\alpha+\beta}(y-a) \mathrm{d}y,$$

$$\overline{Z}^a_{\alpha,\beta}(x) := \overline{Z}_{\alpha+\beta}(x-a) - \beta \int_0^x W_\alpha(x-y) \overline{Z}_{\alpha+\beta}(y-a) \mathrm{d}y.$$

Recall

- The Parisian reflected process is

$$X_r(t) = X(t) + R_r(t), \quad t \geq 0,$$

with the cumulative amount of (Parisian) reflection until time $t \geq 0$:

$$R_r(t) := \sum_{T_0^-(i) \leq t} |X_r(T_0^-(i)-)|, \quad t \geq 0.$$

- Absolute ruin time:

$$\tau_a^-(r) := \inf \{t > 0 : X_r(t) < a\}, \quad a < 0,$$

- Up-crossing time:

$$\tau_b^+(r) := \inf \{t > 0 : X_r(t) > b\}, \quad b > 0.$$

Joint Laplace transform with killing

- For all $q, \theta \geq 0$, $a < 0 < b$, and $x \leq b$,

$$\mathbb{E}_x \left(e^{-q\tau_b^+(r) - \theta R_r(\tau_b^+(r))}; \tau_b^+(r) < \tau_a^-(r) \right) = \frac{\mathcal{H}_{q,r}^a(x,\theta)}{\mathcal{H}_{q,r}^a(b,\theta)},$$

$$\mathbb{E}_x \left(e^{-q\tau_a^-(r) - \theta R_r(\tau_a^-(r))}; \tau_a^-(r) < \tau_b^+(r) \right) = \mathcal{I}_{q,r}^a(x) - \frac{\mathcal{H}_{q,r}^a(x,\theta)}{\mathcal{H}_{q,r}^a(b,\theta)} \mathcal{I}_{q,r}^a(b),$$

where, for $y \in \mathbb{R}$,

$$\mathcal{H}_{q,r}^a(y,\theta) := r \int_0^{-a} e^{-\theta u} \left[W_a^{q,r}(y) \frac{W^{(q+r)}(u)}{W^{(q+r)}(-a)} - W_{-u}^{q,r}(y) \right] du + \frac{W_a^{q,r}(y)}{W^{(q+r)}(-a)},$$

$$\mathcal{I}_a^{q,r}(y) := Z_a^{q,r}(y) - W_a^{q,r}(y) \frac{Z^{(q+r)}(-a)}{W^{(q+r)}(-a)}.$$

- The cases $a \downarrow -\infty$ and $b \uparrow \infty$ admit simpler expression.

Special case: absolute ruin probability

- When $\Phi_0 > 0$ (or X drifts to $-\infty$), then $\mathbb{P}_x\left(\tau_a^-(r) < \infty\right) = 1$.
- When $\Phi_0 = 0$ (or X drifts to ∞ or oscillates),

$$\mathbb{P}_x\left(\tau_a^-(r) < \infty\right) = 1 - \kappa'(0+)\frac{1 - \mathcal{I}_{0,r}^a(x)}{\mathcal{G}_{0,r}(-a,0)},$$

where

$$\mathcal{G}_{0,r}(y,0) := \frac{Z_r(y)^2}{W_r(y)} - r\overline{Z}_r(y).$$

Total discounted capital injections with killing

- For $a < 0 < b$, $q \geq 0$, and $x \leq b$,

$$\mathbb{E}_x \left(\int_0^{\tau_b^+(r) \wedge \tau_a^-(r)} e^{-qt} dR_r(t) \right) = \frac{\mathcal{H}^a_{q,r}(x,0)}{\mathcal{H}^a_{q,r}(b,0)} h^a_{q,r}(b) - h^a_{q,r}(x),$$

where, for $y \in \mathbb{R}$,

$$h^a_{q,r}(y) := \frac{r}{q+r} \Big(\overline{Z}^{(q)}(y) + \frac{aZ^{(q+r)}(-a) + \overline{Z}^{(q+r)}(-a)}{W^{(q+r)}(-a)} W_a^{q,r}(y) - aZ_a^{q,r}(y) - \overline{Z}_a^{q,r}(y) \Big).$$

- The cases $a \downarrow -\infty$ and $b \uparrow \infty$ admit simpler expression.

Recall

- The Parisian reflected process with additional classical reflection above is

$$Y_r^b(t) = X(t) + R_r^b(t) - L_r^b(t), \quad t \geq 0.$$

- $R_r^b(t)$: the cumulative amounts of Parisian reflection from below until time t.
- $L_r^b(t)$: the cumulative amounts of classical reflection from above until time t.
- Absolute ruin time:

$$\eta_a^-(r) := \inf\{t > 0 : Y_r^b(t) < a\}, \quad a < 0.$$

Joint Laplace transform

- Fix $a < 0 < b$, $q \geq 0$, and $\theta \geq 0$. For all $x \leq b$,

$$\mathbb{E}_x \left(e^{-q\eta_a^-(r) - \theta R_r^b(\eta_a^-(r))} \right) = \mathcal{I}_{q,r}^a(x) - \frac{\mathcal{H}_{q,r}^a(x,\theta)}{\mathcal{H}_{q,r}^{a\prime}(b,\theta)} \mathcal{I}_{q,r}^{a\prime}(b),$$

- $\eta_a^-(r) < \infty$ a.s.

Total discounted dividends with killing

- For $a < 0 < b$ and $q \geq 0$, we have

$$\mathbb{E}_x \left(\int_0^{\eta_a^-(r)} e^{-qt} dL_r^b(t) \right) = \begin{cases} \mathcal{H}_{q,r}^{a}(x,0)/\mathcal{H}_{q,r}^{a\prime}(b,0) & x \leq b, \\ \mathcal{H}_{q,r}^{a}(b,0)/\mathcal{H}_{q,r}^{a\prime}(b,0) + (x-b) & x > b. \end{cases}$$

- The case $a \downarrow -\infty$ admits simpler expression.

Total discounted capital injections with killing

- Suppose $q \geq 0$ and $a < 0 < b$. We have

$$\mathbb{E}_x \left(\int_0^{\eta_a^-(r)} e^{-qt} dR_r^b(t) \right) = \begin{cases} \frac{\mathcal{H}_{q,r}^{a}(x,0)}{\mathcal{H}_{q,r}^{a\prime}(b,0)} h_{q,r}^{a\prime}(b) - h_{q,r}^{a}(x) & x \leq b, \\ \frac{\mathcal{H}_{q,r}^{a}(b,0)}{\mathcal{H}_{q,r}^{a\prime}(b,0)} h_{q,r}^{a\prime}(b) - h_{q,r}^{a}(b) & x > b. \end{cases}$$

- The case $a \downarrow -\infty$ admits simpler expression.

Optimization problems

Optimality of periodic barrier dividend strategy:

- Dual model (spectrally positive case) has been solved in Pérez & Yamazaki (2016) – direct application of this talk.
- Primal model (spectrally negative case) – probably difficult, but maybe doable if we assume that the Lévy measure has a completely monotone density.

Minimization of ruin cost & observation costs (Joint w/ Junca and Pérez)

- minimize the sum of
  - ruin cost (as a function of $\tau_a^-(r)$),
  - capital injection cost (as a function of R),
  - observation cost (as a function of r).

References

Avram, F., Pérez, J.L., and Yamazaki, K. Spectrally negative Lévy processes with Parisian reflection below and classical reflection above. *arXiv:*1604.01436, (2016).

Pérez, J.L. and Yamazaki, K. Periodic barrier strategies for a spectrally positive Lévy process. *arXiv:* arXiv:1604.07718, (2016).

Pérez, J.L. and Yamazaki, K. Mixed Periodic-classical barrier strategies for Lévy risk processes. *arXiv:* arXiv:1609.01671, (2016).