Default Contagion and Systemic Risk in the Presence of Credit Default Swaps

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The Fifth Asian Quantitative Finance Conference

The full paper is available at http://ssrn.com/abstract=2853258.

Plan of Talk

Introduction

Background

- CDS and other credit derivatives are blamed as a major cause of the financial crisis in 2008.

  > $\cdots$, *in trying to understand the credit crisis, many observers have identified credit default swaps to be a prominent villain (Stulz, 2010).*

- The network linkage in financial markets through CDS transactions is said to have amplified the crises.
- Research motivation: to investigate how the cross-holdings of CDSs affect financial stability.

Literature review

- Theoretical papers:
  - Eisenberg and Noe (2001), Suzuki (2002); debt cross-holdings, no default cost.
  - Rogers and Veraart (2013); with default costs
  - Fischer (2014): no default cost, with seniority structure of debts.
- Diffuculty to introduce CDSs: non-monotonicity of payoffs.
  $\Rightarrow$Existence of clearing payment vector seems hard to show.
  - No default cost (Suzuki, 2002; Fischer, 2014, El Bitar et al. 2016): Banach's fixed-point theorem $=$ contraction mapping.
  - With default cost (Rogers and Veraart, 2013): Tarski's fixed point theorem $=$ monotone convergence for bounded sequence.

Summary of our paper (1)

Our model:

- Introduce CDS cross-holdings with default costs into Fischer (2014).
- Propose the fictitious default algorithm with financial covenants, reflecting *technical defaults* observed in actual markets (Kusnetsov and Veraart, 2016).
    - Debt service default: the borrower cannot make a scheduled payment.
    - Technical default: another condition such as safety covenant is violated.

	Mean	5%	95%	N
Market assets/Face debt	0.660	0.303	1.221	148

(Davydenk, 2012)

Summary of our paper (2)

Major results:

- Prove the existence of clearing payment vector under the assumption of our algorithm.
- Show with numerical examples that CDS cross-holdings can have a negative impact on financial stability.
  - Cross-ownership of debts; complete graph leads to a more stabile market (Allen and Gale, 2000).
  - Cross-ownership of CDSs; complete graph leads to a default contagion.

Model Setup

Notations

The following notations are used in this talk:

lower	x, y etc.	scalars
lower and bold	$\mathbf{x}$, $\mathbf{y}$, etc.	vectors
upper and bold	$\mathbf{X}$, $\mathbf{Y}$, etc.	matrices

$$\mathbf{0} = (0,\ldots,0)^\top, \quad \mathbf{1} = (1,\ldots,1)^\top, \quad \mathbf{I} = \begin{pmatrix} 1 & & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & & 1 \end{pmatrix},$$

$$\mathbf{x} \wedge \mathbf{y} = \begin{pmatrix} \min\{x_1, y_1\} \\ \vdots \\ \min\{x_n, y_n\} \end{pmatrix}, \quad \mathbf{x} \vee \mathbf{y} = \begin{pmatrix} \max\{x_1, y_1\} \\ \vdots \\ \max\{x_n, y_n\} \end{pmatrix}, \quad (\mathbf{x})^+ = \mathbf{x} \vee \mathbf{0},$$

Banks, banks' business assets, and default costs

- One-shot economy with current and maturity times.
- There are totally n banks in the financial market.
- Each bank has its own business (external) asset.
- $\mathbf{e} = (e_1, \ldots, e_n)^\top \in \boldsymbol{R}^n_+$ denotes the vectors of banks' business asset at maturity before default procedure.
- If bank i defaults, then its business asset e_i is reduced to $(1 - c_i)e_i$, where c_i is a constant. We write
$$\mathbf{C} = \text{diag}(\{c_i\}_{i=1}^n).$$
- Define an $n \times n$ diagonal matrix of default indicators:
$$\boldsymbol{\Delta} = \text{diag}(\{1_{\mathscr{D}}(i)\}_{i=1}^n), \quad 1_{\mathscr{D}}(i) = \begin{cases} 1 & \text{if } i \in \mathscr{D}, \\ 0 & \text{if } i \notin \mathscr{D}, \end{cases}$$
where $\mathscr{D} = \{i \in \{1, \ldots, n\} |$ bank i defaults$\}$.
- banks' business asset value with default costs are described as
$$(\mathbf{I} - \mathbf{C}\boldsymbol{\Delta})\mathbf{e}.$$

Financial securities in the market

- Equities.
- Straight debts:
  - Bank k issues a straight debt with face value $\bar{p}^k$.
- Credit Default Swaps:
  - Bank j writes a CDS with reference bank k.
  - λ_{jk}: the ratio of total CDS issuance to the face value $\bar{p}^k$.
    - If bank k defaults and the payoff of its straight debt is p_k^k, bank j needs to repay $\lambda_{jk}(\bar{p}^k - p_k^k)$ in total.
  - Contractual repayment should be $d_j^k(p_k^k) = \max\left\{\lambda_{jk}(\bar{p}^k - p_k^k), 0\right\}$ (default put option).
  - d_j^k is not increasing in p_k^k.

Clearing payments (payoffs)

- Equities:
  - p_k^0 denotes the final payoff of bank k's equity.
- Straight debts:
  - p_k^k denotes the final payoff of bank k's debt with face value $\bar{p}_k$.
- CDSs:
  - p_j^k denotes the final payoff of d_j^k, the CDS written by j with reference bank k.
- Write $\mathbf{p}^k = \left(p_1^k, p_2^k, \ldots, p_n^k\right)^\top \in \boldsymbol{R}^n$ and define payment vector in the market:

$$\mathbf{p} = \left((\mathbf{p}^0)^\top, (\mathbf{p}^1)^\top, \ldots, (\mathbf{p}^n)^\top\right)^\top \in \boldsymbol{R}^{n(n+1)}$$

Total assets

- Banks cross-hold debts, equities, CDSs issued by other banks.
- Denote by m_{ij}^k bank i's proportion of ownership of the CDS issued by bank j with reference on bank k.
  - Bank i has a right to receive $m_{ij}^k \max\left\{\lambda_{jk}(\bar{p}_k - p_k^k), 0\right\}$ from bank j.
- The ownership structure in the interbank market can be written by

$$\mathbf{M}^k = \left(m_{ij}^k\right)_{i,j=1,\dots,n} \text{ for } k = 0, \dots, n.$$

- Note that
  - $\mathbf{M}^0$ means equity ownership structure.
  - $\mathbf{M}^k$ includes debt ownership structure, $m_{ik}^k, i, k = 1, \dots, n$.
- The total assets of each bank are written by

$$\mathbf{a}(\mathbf{p}; \mathbf{\Delta}) = (\mathbf{I} - \mathbf{C}\mathbf{\Delta})\mathbf{e} + \sum_{k=0}^{n} \mathbf{M}^k \mathbf{p}^k.$$

Contract payoff

- $\mathbf{d}^k(\mathbf{p})$: contract payment function of debts and CDSs with reference on bank k.
    - Contract debt payment functions are given by
    $$d_j^j(\mathbf{p}) = \bar{p}^j, \quad j = 1, \ldots, n.$$
    - Contract CDS payment functions are given by
    $$d_j^k(\mathbf{p}) = \lambda_{jk}\left(\bar{p}^k - p_k^k\right)^+, k \neq j$$
    where the face values' structure of CDSs
    $$\mathbf{\Lambda} = \left(\lambda_{jk}\right)_{j,k=1,\ldots,n}$$
    represents bank j's issuing proportion of CDS to the face value of bank k's debt.

Sub-senior structure of liabilities

- Define $\phi_j(k) \in \{1, \ldots, n\}$ to be the order function of repayment of bank j with reference on k, where

$$
\begin{aligned}
\phi_j(k_1) = 1 &\quad \Leftrightarrow \quad \text{bank } j \text{ repays } p_j^{k_1} \text{ first,} \\
\phi_j(k_2) = 2 &\quad \Leftrightarrow \quad \text{bank } j \text{ repays } p_j^{k_2} \text{ second,} \\
&\quad \vdots \\
\phi_j(k_n) = n &\quad \Leftrightarrow \quad \text{bank } j \text{ repays } p_j^{k_n} \text{ last.}
\end{aligned}
$$

- The sum of bank j's repayment that is senior to ($d_j^k(\mathbf{p})$) is give by

$$\bar{d}_j^k(\mathbf{p}) = \sum_{\phi_j(k') < \phi_j(k)} d_j^{k'}(\mathbf{p}).$$

- Total amount of bank j's liabilities that are senior to equity is written as

$$\bar{d}_j^0(\mathbf{p}) = \sum_{k=1}^{n} d_j^k(\mathbf{p}).$$

Clearing payment and default

Definition 1

The vector $\mathbf{p}$ *and matrix* $\boldsymbol{\Delta}$ *are a clearing payment vector and a clearing default matrix, respectively, if*

$$\mathbf{p}^0 = \left(\mathbf{a}(\mathbf{p};\boldsymbol{\Delta}) - \bar{\mathbf{d}}^0(\mathbf{p})\right)^+,$$
$$\mathbf{p}^k = \left(\mathbf{a}(\mathbf{p};\boldsymbol{\Delta}) - \bar{\mathbf{d}}^k(\mathbf{p})\right)^+ \wedge \mathbf{d}^k(\mathbf{p}).$$

The pair $(\mathbf{p}, \boldsymbol{\Delta})$ *is said to be a clearing system.*

- The equation system in Definition 1 can be expressed as
$$\mathbf{p} = \mathbf{f}(\mathbf{p};\boldsymbol{\Delta}).$$
- The function $\mathbf{f}$ reflects the limited liability and priority rule as in Fischer (2014).

Existence of Clearing System

Fictitious default algorithm with financial covenants

Assumption 1 (Fictitious default algorithm with financial covenants)

At maturity, the clearing default matrix is determined in the following way.

0. *Set* $\mathbf{\Delta}^{(0)} = \mathbf{O}$.
1. *For the first step:*
   (i) *Calculate* $\mathbf{p}^{(1)}$ *satisfying* $\mathbf{p}^{(1)} = \mathbf{f}(\mathbf{p}^{(1)}; \mathbf{\Delta}^{(0)})$.
   (ii) *Set* $\mathscr{D}^{(1)} = \{i \in \{1,\ldots,n\} | p_i^0 = 0\}$, *and*
   (iii) *Update* $\mathbf{\Delta}^{(1)} = \mathrm{diag}(\{1_{\mathscr{D}^{(1)}}(i)\}_{i=1}^n)$.
2. *For the* ℓ*-th step:*
   (i) *Calculate* $\mathbf{p}^{(\ell)}$ *satisfying* $\mathbf{p}^{(\ell)} = \mathbf{f}(\mathbf{p}^{(\ell)}; \mathbf{\Delta}^{(\ell-1)})$.
   (ii) *Set* $\mathscr{D}^{(\ell)} = \mathscr{D}^{(\ell-1)} \cup \{i \in \{1,\ldots,n\} | p_i^0 = 0\}$.
   (iii) *Update* $\mathbf{\Delta}^{(\ell)} = \mathrm{diag}(\{1_{\mathscr{D}^{(\ell)}}(i)\}_{i=1}^n)$.
3. *Stop when* $\mathbf{\Delta}^{(\ell)} = \mathbf{\Delta}^{(\ell-1)}$ *and set* $(\mathbf{p}, \Delta) = (\mathbf{p}^{(\ell)}, \Delta^{(\ell)})$.

Some remarks on our algorithm

- Our algorithm coincides with a generalised clearing vector in Kusnetsov and Veraart (2016).
- A natural assumption for default is
$$\mathscr{D} = \{i \in \{1,\dots,n\} | p_i^0 = 0\}$$
with a clearing payment vector $\mathbf{p}$ as in Elsenberg and Noe (2001) and other related studies.
- Under Assumption 1, the above result does not necessarily hold and it can be that $i \in \mathscr{D}$ and $p_i^0 > 0$ at the same time.
    - Once a bank is taken as default in the sequential procedure, it should incur default costs and cannot be solvent for ever. $\Rightarrow$Techincal default.
- $\mathscr{D}^{(\ell)}$ is increasing in ℓ.
    - There always exists a clearing system if we have a vector $\mathbf{p}$ such that $\mathbf{p} = \mathbf{f}(\mathbf{p};\boldsymbol{\Delta})$ for any $\boldsymbol{\Delta}$.

Contraction mapping

Assumption 2

$$\sum_{i=1}^{n} m_{ij}^{k} < 1$$

for $k = 0, \dots, n$.

Lemma 1

For any given $\boldsymbol{\Delta}$*, the equation system* $\mathbf{p} = \mathbf{f}(\mathbf{p}; \boldsymbol{\Delta})$ *has a unique solution under Assumption 2.*

Sketch of the Proof: For a fixed $\boldsymbol{\Delta}$, $\mathbf{f}$ is continuous. Further under Assumption 2, the mapping $\mathbf{f}$ is contractive in l^1-norm as shown by Fischer (2014). Therefore we can apply Banach's fixed point theorem. □

Main theorem

Theorem 1

There exists a uniquie clearing system under Assumptions 1 and 2.

Proof: The theorem easily follows from Lemma 1 and the monotonicity of $\boldsymbol{\Delta}^{(\ell)}$. □

Remark 1

- *The result on existence does not depend on* $\boldsymbol{\Lambda}$*, issuing structure of CDSs.*
- *Banks can issue leveraged CDSs on other banks in our setting. In other words, we do not need to impose the condition* $\lambda_{ij} < 1$.

Numerical Examples

Simulations

1. Suppose a multivariate Merton (1974) model.
$$e_i = e_{i0} \exp\left\{ \left(\mu - \frac{\sigma^2}{2} \right) + \sigma \varepsilon_i \right\}.$$
$\mathrm{Corr}[\varepsilon_i, \varepsilon_j] = \rho$ for $i \neq j$.
2. Conduct Monte Carlo simulations to get
$$\tilde{\mathbf{e}}^{(h)} = (\tilde{e}_1^{(h)}, \ldots, \tilde{e}_n^{(h)})^\top \text{ for } h = 1, \ldots, \eta,$$
where η is the number of simulations.
3. For each $\tilde{\mathbf{e}}^{(h)}$, obtain the clearing system $(\mathbf{p}, \boldsymbol{\Delta})$ with our algorithm.
4. Calculate the probability
$$\mathbb{P}\{\text{\# of defaulted banks is } \xi\}$$
for $\xi = 0, 1, \ldots, n$.

Cross-ownership structure (1)

Three types of financial markets are considered.

- Type A: No cross-ownership.

$$e_{i0} = \bar{g},$$
$$m^k_{ij} = 0.$$

- Type B-ℓ_1: Cross-ownership of debts with ℓ_1 banks for $\ell_1 = 0, 1, \ldots, n-1$.

$$e_{i0} = \bar{g} - \ell_1/n,$$
$$\lambda_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases}$$
$$m^k_{ij} = \begin{cases} \frac{1}{n} & \text{if } i \in [1, n-\ell_1], j \in [i+1, i+\ell_1], \text{ and } j = k, \\ & \text{if } i \in [n-\ell_1+1, n-1], j \in [1, i+\ell_1-n] \cup [i+1, n], \text{ and } j = k, \\ & \text{if } i = n, j \in [1, \ell_1], \text{ and } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Cross-ownership structure (2)

- Type C-(ℓ_1, ℓ_2): Cross-ownership of debts among ℓ_1 banks and CDSs among ℓ_2 banks for $\ell_1 = 0, 1, \ldots, n-1$ and $\ell_2 = 0, 1, \ldots, n-2$.

$$e_{i0} = \bar{g} - \ell_1 / n,$$

$$\lambda_{jk} = \begin{cases} 1 & \text{if } j = k, \\ \frac{1}{n} & \text{if } j \in [1, n-\ell_2] \text{ and } k \in [j+1, j+\ell_2], \\ & \text{if } j \in [n-\ell_2+1, n-1] \text{ and } k \in [1, j+\ell_2-n] \cup [j+1, n], \\ & \text{if } j = n \text{ and } k \in [1, \ell_2], \\ 0 & \text{otherwise,} \end{cases}$$

$$\Phi = \{\phi_j(k)\}_{j,k=1}^{n} = \begin{pmatrix} 1 & 2 & 3 & \ldots & n-1 & n \\ n & 1 & 2 & \ldots & n-2 & n-1 \\ \vdots & & & \ddots & & \vdots \\ 2 & 3 & 4 & \ldots & n & 1 \end{pmatrix},$$

Cross-ownership structure (3)

- Type C-(ℓ_1, ℓ_2): Cross-ownership of debts among ℓ_1 banks and CDSs among ℓ_2 banks for $\ell_1 = 0, 1, \dots, n-1$ and $\ell_2 = 0, 1, \dots, n-2$ (cont.).

$$
m_{ij}^k \begin{cases} \frac{1}{n} & \text{if } i \in [1, n-\ell_1], j \in [i+1, i+\ell_1], \text{ and } j = k, \\ & \text{if } i \in [n-\ell_1+1, n-1], j \in [1, i+\ell_1-n] \cup [i+1, n], \text{ and } j = k, \\ & \text{if } i = n, j \in [1, \ell_1], \text{ and } j = k, \\ & \text{if } i \in [1, n-\ell_2-1], j \in [i+1, i+\ell_2], \text{ and } k = j+1, \\ & \text{if } i = n-\ell_2, j \in [i+1, n-1], \text{ and } k = j+1, \\ & \text{if } i = n-\ell_2, j = n, \text{ and } k = 1, \\ & \text{if } i \in [n-\ell_2+1, n-1], j \in [1, i+\ell_2-n], \text{ and } k = j+1, \\ & \text{if } i \in [n-\ell_2+1, n-1], j = n, \text{ and } k = 1, \\ & \text{if } i \in [n-\ell_2+1, n-1], j \in [i+1, n-1], \text{ and } k = j+1, \\ & \text{if } i = n, j \in [1, \ell_2], \text{ and } k = j+1, \\ 0 & \text{otherwise.} \end{cases}
$$

Parameter values

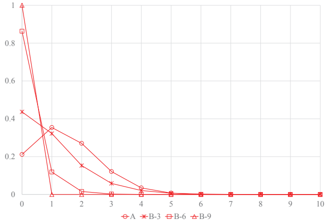
- Basecase parameters:

n	number of banks	10
$\bar{p}^i$	face value of debts	1
$\bar{b}$	initail value of asset	0.2
c_i	default cost ratio	0.5
μ	growth rate of asset	0.05

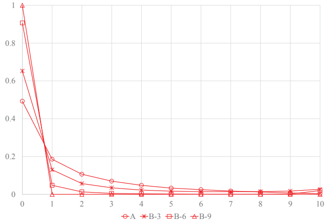
- Four cases are considered:
  - Asset volatility $\sigma = 0.2$ or 0.5.
  - Asset correlation $\rho = 0$ or 0.5.
- Number of simulations: $\eta = 100,000$.

Default Probabilities (1)

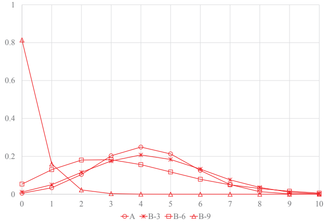
Type B (cross-ownership of debts):



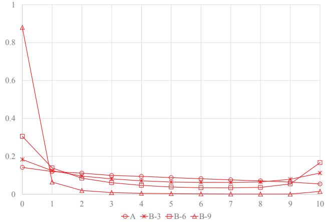
$(\sigma, \rho) = (0.2, 0.0)$



$(\sigma, \rho) = (0.2, 0.5)$



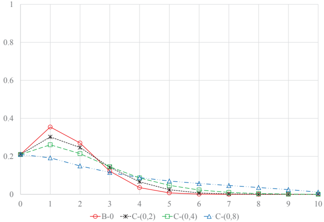
$(\sigma, \rho) = (0.5, 0.0)$



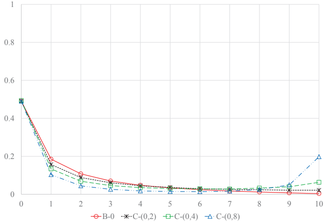
$(\sigma, \rho) = (0.5, 0.5)$

Default Probabilities (2)

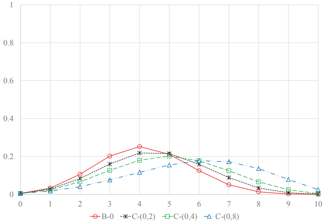
Type C-$(0, \ell_2)$ (no cross-ownership of debts):



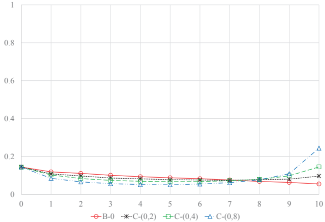
$(\sigma, \rho) = (0.2, 0.0)$



$(\sigma, \rho) = (0.2, 0.5)$



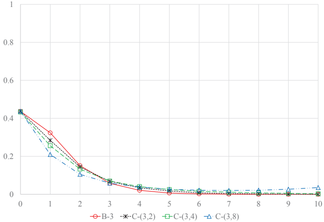
$(\sigma, \rho) = (0.5, 0.0)$



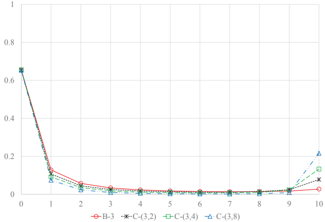
$(\sigma, \rho) = (0.5, 0.5)$

Default Probabilities (3)

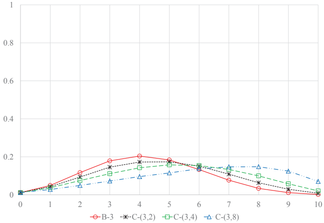
Type C-$(3, \ell_2)$ (cross-ownership of debts among 3 banks):



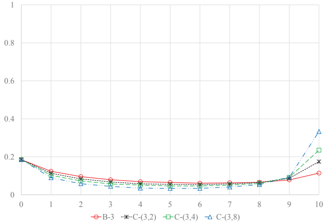
$(\sigma, \rho) = (0.2, 0.0)$



$(\sigma, \rho) = (0.2, 0.5)$



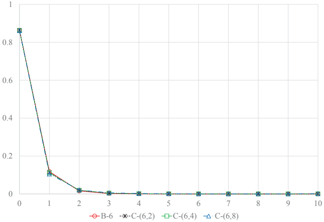
$(\sigma, \rho) = (0.5, 0.0)$



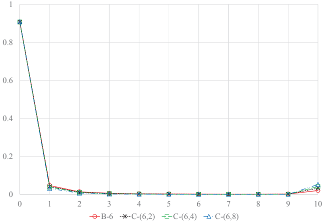
$(\sigma, \rho) = (0.5, 0.5)$

Default Probabilities (4)

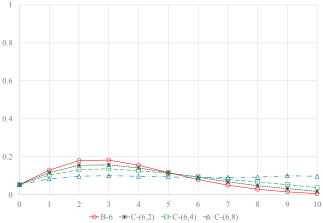
Type C-$(6, \ell_2)$ (cross-ownership of debts among 6 banks):



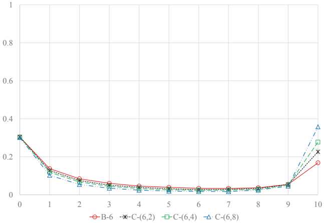
$(\sigma, \rho) = (0.2, 0.0)$



$(\sigma, \rho) = (0.2, 0.5)$



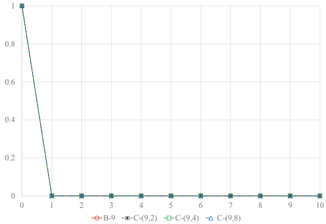
$(\sigma, \rho) = (0.5, 0.0)$



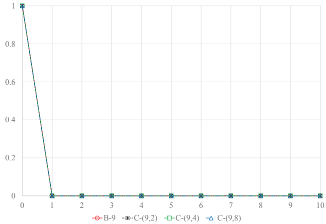
$(\sigma, \rho) = (0.5, 0.5)$

Default Probabilities (5)

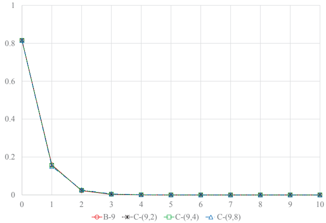
Type C-$(9, \ell_2)$ (cross-ownership of debts among 9 banks):



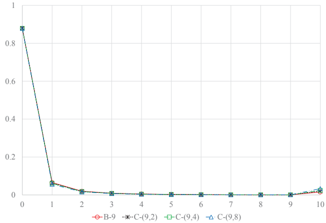
$(\sigma, \rho) = (0.2, 0.0)$



$(\sigma, \rho) = (0.2, 0.5)$



$(\sigma, \rho) = (0.5, 0.0)$



$(\sigma, \rho) = (0.5, 0.5)$

Major observation

- The effect of cross-ownerships:
  - Debts; more stable financial market (Allen and Gale, 2000; Acemuglu et al., 2015).
  - CDSs; unstable financial market with default contagion.
- This is the first study to show that strong connectedness (complete graph) may lead to market vulnerability and increase systemic risk.

Conclusion

Conclusion

- Extended Fischer (2014) to the model with cross-holdings of CDS as well as banks' default costs and focus on CDS market.
- Proposed fuctitious default algorithm with financial covenants.
- Proved existence theorem for clearing system.
- Showed with numerical examples that the cross-holdings of CDS increase the systemic risk of financial markets.

References I

[] Acemuglu, D., A. Ozdaglar, and A. Tahbaz-Salehi (2015), “Systemic Risk and Stability in Financial Networks," *American Economic Review*, **105**(2), 564–608.

[1] Allen, F. and D. Gale (2000), “Financial Contagion," *Journal of Political Economy*, **108**(1), 1–33.

[2] Davydenko, S. A. (2012), “When Do Firms Default? A Study of the Default Boundary,” working paper.

[3] Eisenberg, L. and T. H. Noe (2001), “Systemic Risk in Financial Systems,” *Management Science*, **47**(2), 236–249.

[4] El Bitar, K., Y. Kabanov, and R. Mokbel (2016), “On Uniqueness of Clearing Vectors Reducing the Systemic Risk," forthcoming in *Informatics and Applications*.

[5] Fischer, T. (2014), “No-Arbitrage Pricing under Systemic Risk: Accounting for Cross- Ownership,” *Mathematical Finance*, **24**, 94–124.

[6] Kusnetsov, M. and L. A. M. Veraart (2016), “Interbank Clearing in Financial Networks," working paper.

References II

[7] Merton, R. C. (1974), “On the Pricing of Corporate Debt: The Risk Structure of Interest Rates,” *Journal of Finance*, **29**(2), 449–470

[8] Rogers, L. C. G. and Veraart, L. A. M. (2013), “Failure and Rescue in an Interbank Network,” *Management Science*, **59**(4), 882–898

[9] Stulz, R. M. (2010) "Credit Default Swaps and the Credit Crisis,” *Journal of Economic Perspectives*, **24**(1), 73–92.

[10] Suzuki, T. (2002), “Valuing Corporate Debt: The Effect of Cross-Holdings of Stock and Debt,” *Journal of the Operations Research Society of Japan*, **45**(2), 123–144.

Thank you for your attention