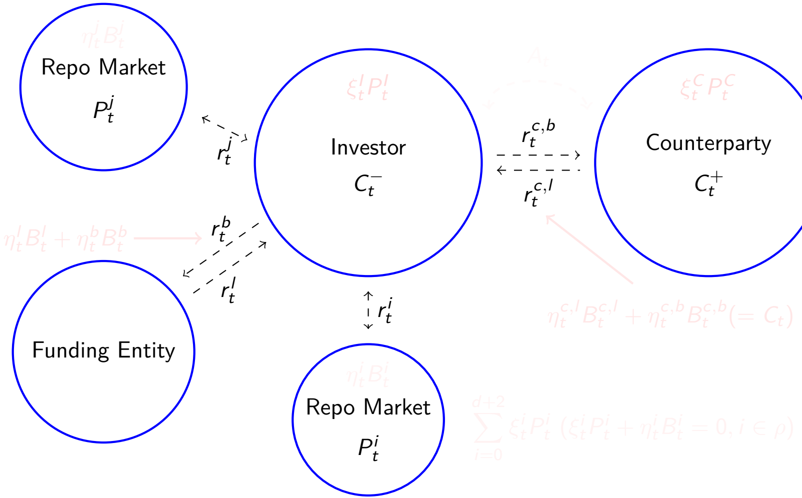
Recovering Linear Equations of XVA in Bilateral Contracts

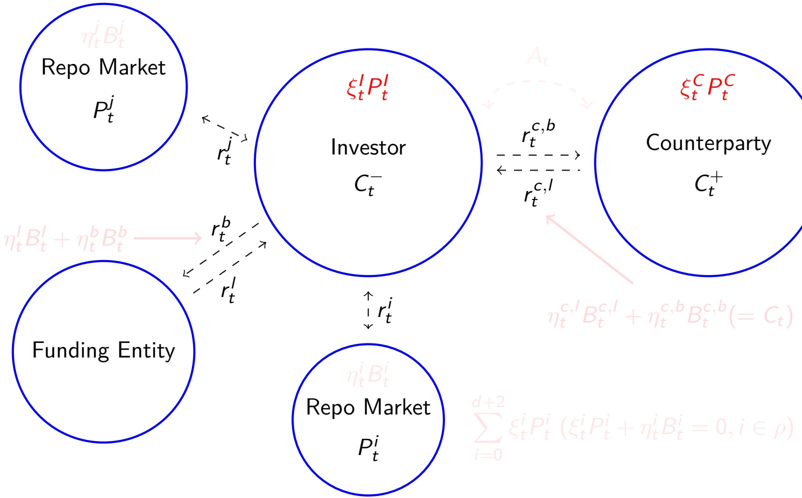
Junbeom Lee, Chao Zhou

Department of Mathematics, National University of Singapore

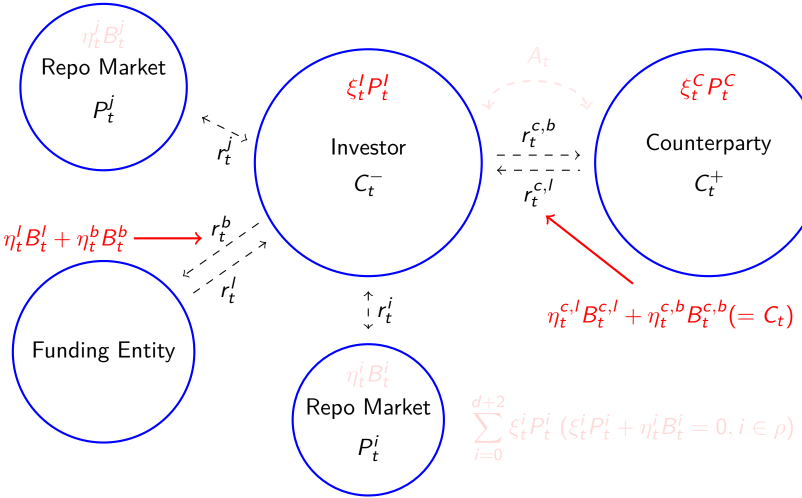
The game changer



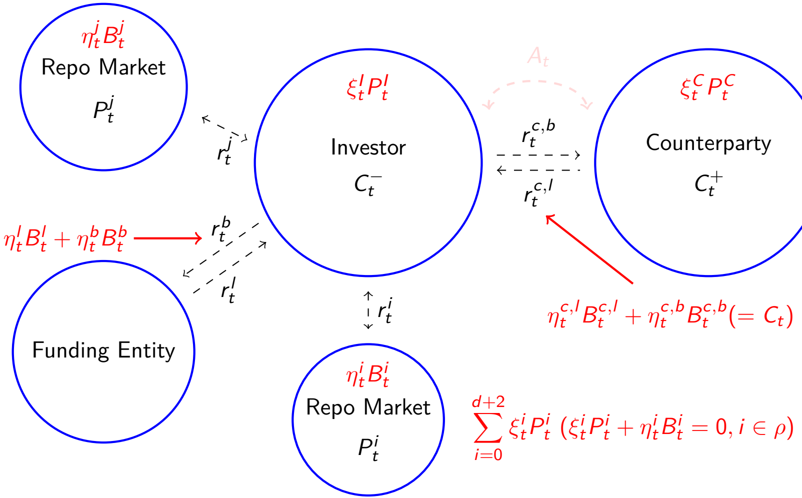
The game changer



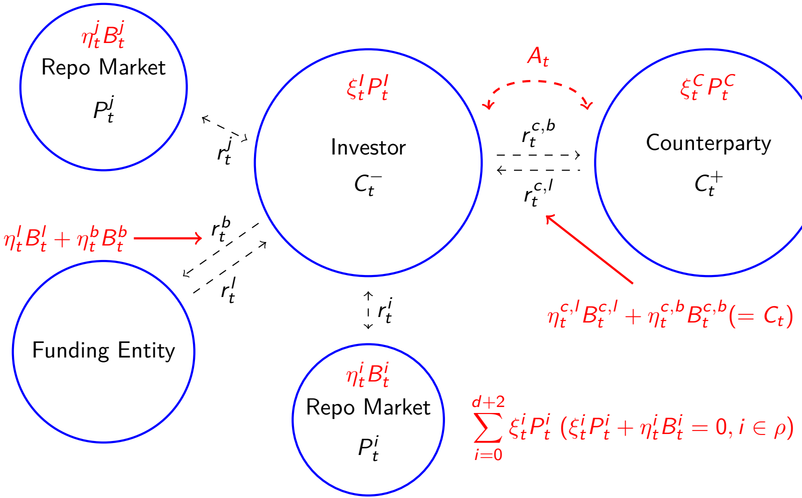
The game changer



The game changer



The game changer



$$dV_t = \sum_{i=0}^{d+2} \xi_t^i dP_t^i + \sum_{i\in\rho} \eta_t^i dB_t^i + C_t^+ r_t^{c,l} dt - C_t^- r_t^{c,b} dt - dA_t$$
$$+ (V_t - C_t - \sum_{i\in\rho^c} \xi_t^i P_t^i)^+ r_t^l dt - (V_t - C_t - \sum_{i\in\rho^c} \xi_t^i P_t^i)^- r_t^b dt,$$

$$\tilde{X}_t = e^{-\int_0^t r_s^D ds} X_t,\ \tilde{\kappa} = e^{-\int_0^T r_s^D ds}\kappa,\ Y_t = V_t - C_t,\ \tau = \tau^I \wedge \tau^C,$$

$$dP_t^i = P_t^i(r_t dt + \sigma_t^i dW_t), i \in \{0, 1, \dots, d\}$$
$$dP_t^i = P_{t-}^i(r_t dt + \sigma_t^i dW_t - dM_t^i), i \in \{I, C\}.$$

$$
\begin{aligned}
-\tilde{Y}_t = & \tilde{C}_t - \int_{t+}^{\bar{\tau}} (B_s^D)^{-1} dA_s + \int_{t+}^{\bar{\tau}} \tilde{f}_s ds + \int_{t+}^{\bar{\tau}} \tilde{m}_s ds \\
& + \sum_{i=0}^{d+2} \int_{t+}^{\bar{\tau}} \xi_s^i \tilde{P}_s^i \sigma_s^i dW_s^{\mathbb{Q}} - \sum_{i \in \{I,C\}} \int_{t+}^{\bar{\tau}} \int_{\mathbb{R}} \xi_s^i \tilde{P}_s^i z \tilde{\mu}^{i,\mathbb{Q}},
\end{aligned}
$$

$$
\begin{aligned}
\tau &= \tau^I \wedge \tau^C, \bar{\tau} = \tau \wedge T \\
A_t &= \mathbb{1}_{(0,\tau)}(t) A_t^c + \mathbb{1}_{\tau \leq T} \mathbb{1}_{[\tau, T]}(t) \theta_\tau \\
\theta_\tau &= \varepsilon_\tau + \mathbb{1}_{t < \tau^C \leq \tau^I \wedge T} L_C (\varepsilon_\tau - C_{\tau-})^- - \mathbb{1}_{t < \tau^I \leq \tau^C \wedge T} L_I (\varepsilon_\tau - C_{\tau-})^+ \\
K_t &= \left(Y_t - \sum_{i \in \rho^c} \xi_t^i P_t^i \right) \\
\tilde{f}_t &= \left(\mathbb{1}_{K_t \geq 0} b_t^l + \mathbb{1}_{K_t < 0} b_t^b \right) \tilde{K}_t - \sum_{i \in \rho} b_t^i \xi_t^i \tilde{P}_t^i \\
\tilde{m}_t &= \tilde{C}_t^+ b_t^{c,l} - \tilde{C}_t^- b_t^{c,b}
\end{aligned}
$$

Why?

Goal) Recovering linear BSDEs by comparison of K_t [El Karoui et al., 1997].

Why?) Given that we realize non-negligible spreads between lending/borrowing rates, r^l_t/r^b_t, the value of a derivative is

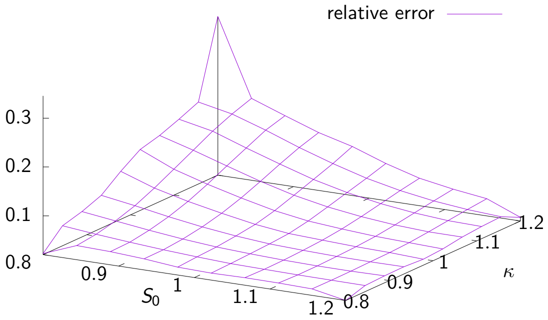
1. explicit solution [Piterbarg, 2010, Bichuch et al., 2015] with the assumption, $r^l_t = r^b_t$ (or $r^f_t = \frac{1}{2}(r^l_t + r^b_t)$)
2. numerical approximation
   - Analytical approximation [Gobet and Pagliarani, 2015]
   - Least Square Monte Carlo

The pitfall: Numercial Test for a Call Option

- Call Option
- least square Monte Carlo simulation,
- basis for conditional expectation is second order polynomial
- # of simulation=10000, $\Delta t = 0.025$
- parameters:

γ	σ	r^l	r^b	r	T
0	0.2	0.001	0.3	0.005	0.5

The pitfall: Numercial Test for a Call Option



Assumption: What are restricted and not restricted

- Stochastic interest rates.
- Non-Markov
- The condition in our result is about $r_t^{c,b}, r_t^{c,l}$ and collateral rule. We do not touch what is given such as r_t^l and r_t^b.
- Deterministic volatility, spreads. Constant hazard rates.
- Immersion Hypothesis
- Complete market
- $\rho = \emptyset$ for interest rate derivatives and $\rho = \{I, C\}$ for others

$$-\tilde{Y}_t = \tilde{C}_t - \int_{t+}^{\bar{\tau}} (B_s^D)^{-1} dA_s + \int_{t+}^{\bar{\tau}} \tilde{f}_s ds + \int_{t+}^{\bar{\tau}} m(s, \tilde{C}_s) ds$$
$$+ \sum_{i=0}^{d+2} \int_{t+}^{\bar{\tau}} \xi_s^i \tilde{P}_s^i \sigma_s^i dW_s^{\mathbb{Q}} - \sum_{i \in \{I,C\}} \int_{t+}^{\bar{\tau}} \int_{\mathbb{R}} \xi_s^i \tilde{P}_{s-}^i z \tilde{\mu}^{i,\mathbb{Q}}(ds, dz)$$

Lemma

1) If $\sigma_t^0 = \sigma_t^I = \sigma_t^C$, $\mathbb{1} \cdot (\sigma_t^*)^{-1} D_t \tilde{Y}_t = \sum_{i=0}^{d+2} \xi_t^i \tilde{P}_t^i$

2) If $\sigma_t^I = \sigma_t^C = 0$, $\mathbb{1} \cdot (\sigma_t^*)^{-1} D_t \tilde{Y}_t = \sum_{i=0}^{d} \xi_t^i \tilde{P}_t^i$.

3) For $i \in \{I, C\}$, $-D_{i,t,1} \tilde{Y}_t = \xi_t^i \tilde{P}_t^i$.

Comparison

$$
\begin{aligned}
-\tilde{Y}_t =& J_t + \int_{t+}^{\bar{\tau}} (b_s^l \mathbb{1}_{K_s \geq 0} + b_s^b \mathbb{1}_{K_s < 0}) \left(\tilde{Y}_s - \sum_{i=0}^{d+2} \xi_s^i \tilde{P}_s^i \right) ds \\
&+ \sum_{i=0}^{d+2} \int_{t+}^{\bar{\tau}} \xi_s^i \tilde{P}_s^i \sigma_s^i dW_s^{\mathbb{Q}} - \sum_{i \in \{I,C\}} \int_{t+}^{\bar{\tau}} \int_{\mathbb{R}} \xi_s^i \tilde{P}_s^i z \tilde{\mu}^{i,\mathbb{Q}},
\end{aligned}
$$

$$
\begin{aligned}
-D_t \tilde{Y}_t =& D_t J_t + \int_{t+}^{\bar{\tau}} (b_s^l \mathbb{1}_{K_s \geq 0} + b_s^b \mathbb{1}_{K_s < 0}) \left(D_s \tilde{Y}_s - \sum_{i=0}^{d+2} D_s(\xi_s^i \tilde{P}_s^i) \right) ds \\
&+ \sum_{i=0}^{d+2} \int_{t+}^{\bar{\tau}} D_s(\xi_s^i \tilde{P}_s^i) \sigma_s^i dW_s^{\mathbb{Q}} - \sum_{i \in \{I,C\}} \int_{t+}^{\bar{\tau}} \int_{\mathbb{R}} D_s(\xi_s^i \tilde{P}_s^i) z \tilde{\mu}^{i,\mathbb{Q}}
\end{aligned}
$$

A. $\tilde{\varepsilon}_t = \mathbb{E}(\int_{t+}^{T} B_s^{-1} dA^c | \mathcal{F}_t)$ and $\tilde{C}_t = \gamma \tilde{\varepsilon}_t$, where $0 \leq \gamma \leq 1$

B. $\tilde{\varepsilon}_t = \tilde{V}_t$ and $\tilde{C}_t = \gamma \tilde{V}_t$, where $0 \leq \gamma \leq 1$

Main result: $\tilde{\varepsilon}_t = \mathbb{E}(\int_{t+}^{T} B_s^{-1} dA^c | \mathcal{F}_t)$ and $\tilde{C}_t = \gamma \tilde{\varepsilon}_t$

$$\tilde{J}_t = \tilde{C}_t - \int_{t+}^{\bar{\tau}} (B_s^D)^{-1} dA_s + \int_{t+}^{\bar{\tau}} m(s, \tilde{C}_s) ds. \qquad (1)$$

Proposition

Let $t < \tau$ and $\mathbb{Q}$ be an equivalent martingale measure with B_t^D as the numéraire. Assume

(i) $\tilde{J}_t \in \mathbb{D}^{1,2}$ *for all* $t \in [0, \bar{\tau}]$,

(ii) *for all* $t \in [0, \bar{\tau}]$, $k \in \{0, 1, \ldots, d\}$, $i, j \in \{I, C\}$

$$\mathbb{E}^{\mathbb{Q}} \left(\int_0^{\bar{\tau}} |D_{k,s} \tilde{J}_t|^2 ds \right) < \infty,$$
$$\mathbb{E}^{\mathbb{Q}} \left(\int_0^{\bar{\tau}} |D_{i,s,1} \tilde{J}_t|^2 \lambda^{j,\mathbb{Q}} ds \right) < \infty,$$

Proposition

(iii) $\mathbb{1} \cdot (\sigma_u^*)^{-1} D_u \tilde{J}_t \geq \tilde{J}_t$ *a.s for all* $u \leq t$,

(iv) *If* $\sigma_t^0 = \sigma_t^I = \sigma_t^C$, $\rho = \emptyset$. *Otherwise* $\rho = \{I, C\}$,

(v) *for* $i \in \{I, C\}$, *when* $i \in \rho$, $b_t^i \leq \lambda^{i,\mathbb{Q}}$, *otherwise* $b_t^b \leq \lambda^{i,\mathbb{Q}}$ *(A stronger condition which is easier to check is that* $\tilde{f}(t, y, z, z^I, z^C)$ *is non-decreasing with respect to* z^I *and* z^C*),*

Then $\tilde{Y}_t$ *is a unique solution of linear BSDE:*

$$
\begin{aligned}
-\tilde{Y}_t =& \tilde{J}_t + \int_{t+}^{\bar{\tau}} b_s^b(\tilde{Y}_s - \sum_{i \in \rho^c} \xi_s^i \tilde{P}_s^i) - \sum_{i \in \rho} b_s^i \xi_s^i P_s^i ds \\
&+ \sum_{i=0}^{d+2} \int_{t+}^{\bar{\tau}} \xi_s^i \tilde{P}_s^i \sigma_s^i dW_s^{\mathbb{Q}} - \sum_{i \in \{I, C\}} \int_{t+}^{\bar{\tau}} \int_{\mathbb{R}} \xi_s^i \tilde{P}_{s-}^i z \tilde{\mu}^{i,\mathbb{Q}}(ds, dz).
\end{aligned}
$$

Example: Stock Forward

Example (Stock Forward)

- The assets follow the following dynamics

$$
\begin{aligned}
dS_t &= r_t S_t dt + \sigma S_t dW_t \\
dP_t^I &= r_t P_t^I dt - P_{t-}^I dM_t^I \\
dP_t^C &= r_t P_t^C dt - P_{t-}^C dM_t^C,
\end{aligned}
$$

  where r_t is deterministic and σ is constant,
- $A_t^c = \mathbb{1}_{[T]}(t)(S_T - \kappa)$.
- $\rho = \{I, C\}$
- $r_t^{c,l} \leq r_t = r_t^{c,b}$ (or $\gamma = 0$),
- $r_t^i \leq \lambda^i + r_t$ for $i \in \{I, C\}$.

Example: Interest Rate Swap

Example (Interest rate swap)

- The assets follow:
$$dP^i_t = P^i_t(r_t dt + \sigma^i dW_t), i \in \{0, 1, \dots, d\}$$
$$dP^i_t = P^i_{t-}(r_t dt + \sigma^i dW_t - dM^i_t), i \in \{I, C\}.$$
- $A^c_t = \mathbb{1}_{[T]}(t)\Phi(P_T) = \mathbb{1}_{[T]}(t)(1 - P^d_T - \kappa \sum_{i=1}^{d} \delta_i P^i_T).$
- If short rate process, r_t, will follow the dynamics: $dr_t = \beta(t, r_t)dt + \sigma^r dW_t$, where $\beta : [0, T] \times \mathbb{R} \to \mathbb{R}$ σ^r is assumed to satisfy the condition that
$$1 + \mathbb{1} \cdot (\sigma^*)^{-1} \int_t^T (\sigma^r)^* ds \leq 0, \quad \text{a.s.}$$
$$\left(\text{In Vasicek, } 1 + (\sigma)^{-1} \int_t^T \sigma^r ds = 1 - \frac{a(T-t)}{1 - e^{-a(T-t)}} \leq 0\right)$$

$$
\begin{aligned}
-\tilde{V}_t =& \tilde{J}_t + \int_t^{\bar{\tau}} (\mathbb{1}_{K\geq 0} b_s^l + \mathbb{1}_{K<0} b_s^b) \left(\tilde{V}_s(1-\gamma) - \sum_{i\in\rho} \xi_s^i \tilde{P}_s^i \right) ds \\
& - \int_t^{\bar{\tau}} \sum_{i\in\rho^c} b_s^i \xi_s^i \tilde{P}_s^i ds \\
& + \int_{t+}^{\bar{\tau}} a_s^1 \tilde{V}_s^+ - a_s^2 \tilde{V}_s^- ds + \sum_{i=0}^{d+2} \int_{t+}^{\bar{\tau}} \xi_s^i \tilde{P}_s^i \sigma_s^i dW_s^{\mathbb{Q}},
\end{aligned}
$$

where $\tilde{J}_t = -\int_{t+}^{\bar{\tau}} (B_s^D)^{-1} dA_s^c$, $a_t^1 = \lambda^{I,\mathbb{Q}} L_I(1-\gamma) + b_t^{c,l} - \lambda^{\mathbb{Q}}$ and $a_t^2 = \lambda^{C,\mathbb{Q}} L_C(1-\gamma) + b_t^{c,b} - \lambda^{\mathbb{Q}}$. Note that both a_t^1 and a_t^2 are deterministic.

Lemma

Consider the following BSDE

$$-\tilde{V}_t = \tilde{\zeta} + \int_t^{\bar{\tau}} a_s^1 \tilde{V}_s^+ - a_s^2 \tilde{V}_s^- ds + \int_t^{\bar{\tau}} \tilde{Z}_s dW_s^{\mathbb{Q}}. \quad (2)$$

Assume $\zeta \leq 0$, $\zeta \in \mathbb{L}_T^2$ and a_t^i are processes that makes the generator standard. Then $\tilde{V}_t \geq 0$.

Proof.

Let us consider the following BSDEs

$$-\tilde{V}_t' = \int_t^{\bar{\tau}} a_s^1 \tilde{V}_s'^+ - a_s^2 \tilde{V}_s'^- ds + \int_t^{\bar{\tau}} \tilde{Z}_s' dW_s^{\mathbb{Q}}. \quad (3)$$

Then $(\tilde{V}_t', \tilde{Z}_t') = (0, 0)$ is the unique solution of (3). Therefore, proof is done by applying comparison principle to (2) and (3). □

Example: Put Option on a Stock

Example (Put option on a stock)

We assume

- The assets follow the dynamics:

$$dS_t = r_t^I S_t dt + \sigma S_t W_t^{\mathbb{Q}}$$
$$dP_t^I = r_t^I P_t^I dt - P_{t-}^I dM_t^{I,\mathbb{Q}}$$
$$dP_t^C = r_t^C P_t^C dt - P_{t-}^C dM_t^{C,\mathbb{Q}},$$

  under a new probability, $\mathbb{Q}$.
- $\rho = \{I, C\}$,
- $A_t^c = \mathbb{1}_{[T]}(t)(\kappa - S_T)^+$. Therefore, $\tilde{\varepsilon}_t = \tilde{c}(t, S_t)$
- $r_t^{c,l} \leq r_t^l \leq r_t^b$.

Example: Floating Strike Asian Call Option

Example (Floating strike Asian call option)

- The assets follow the dynamics:

$$dS_t = r^I_t S_t dt + \sigma S_t W^{\mathbb{Q}}_t$$
$$dP^I_t = r^I_t P^I_t dt - P^I_{t-} dM^{I,\mathbb{Q}}_t$$
$$dP^C_t = r^C_t P^C_t dt - P^C_{t-} dM^{C,\mathbb{Q}}_t,$$

under a new probability, $\mathbb{Q}$.

- $A^c_t = \mathbb{1}_{[T]}\Phi(S_T) = \mathbb{1}_{[T]}(t)(S_T - \kappa A(0,T))^+$ where $A(0,T) = \exp\left(\frac{1}{T}\int_0^T \ln(S_u)du\right)$
- $\rho = \{I, C\}$
- $r^{c,I}_t \leq r^I_t \leq r_t \leq r^b_t$.

REMARK: Extensibility and Weakness

Remark

- *(Carr-Madan decomposition) For any* $f \in \mathcal{C}^2(\mathbb{R})$

$$f(S) = f(\kappa) + f'(\kappa)(S-\kappa) + \int_{-\infty}^{\kappa} f''(K)(K-S)dK + \int_{\kappa}^{\infty} f''(K)(S-K)dK$$

- *If a function* f *has a bounded Hessian on* $A \subset \mathbb{R}^d$*, it can be decomposed into the sum of a convex function and a concave function. [Yuille et al., 2002]*

Thank you!

Bichuch, M., Capponi, A., and Sturm, S. (2015). Arbitrage-free pricing of xva–part i: Framework and explicit examples. *Available at SSRN*.

El Karoui, N., Peng, S., and Quenez, M. C. (1997). Backward stochastic differential equations in finance. *Mathematical finance*, 7(1):1–71.

Gobet, E. and Pagliarani, S. (2015). Analytical approximations of bsdes with nonsmooth driver. *SIAM Journal on Financial Mathematics*, 6(1):919–958.

Piterbarg, V. (2010). Funding beyond discounting: collateral agreements and derivatives pricing. *Risk*, 23(2):97.

Yuille, A. L., Rangarajan, A., and Yuille, A. (2002). The concave-convex procedure (cccp). *Advances in neural information processing systems*, 2:1033–1040.