A Market Driver Volatility Model via Policy Improvement Algorithm

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Introduction

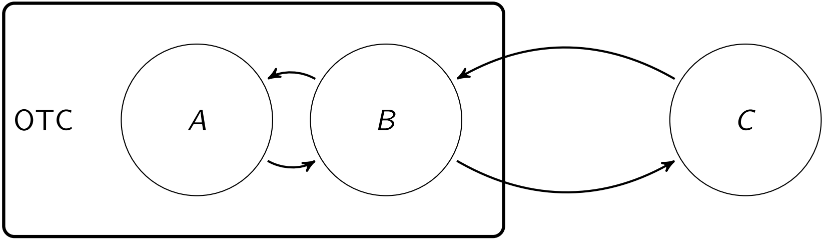
Stochastic volatility models are essential nowadays due to the higher demand of more complex derivatives products (e.g. path-dependent options).
However, if there is a concentration in traders' positions, it causes a change in the volatility dynamics.

Our Aim:

- To come up with a new model that models well the concentration effect
- To come up with efficient approximation algorithm for the price of concentrated product

Concentration

Traders trade volatility in the OTC (Over-the-Counter) market.
Generally, both the buyers and sellers of volatility are traders in the OTC market, hence the total amount of vega (risk of volatility) is conserved.
However, if traders trade with clients who are outside of the OTC market, the vega is no longer conserved.
If traders keep trading the same structure with clients, then the position gets bigger and bigger in the OTC market, hence is the concentration of risk.



Example

Ex. In case Buyer A and Seller B trade 2Y 120 call

Trader	Trade	Hedge	Total
Trader A	+1	0	+1
Trader B	-1	0	-1
Market	**0**	**0**	**0**

Example

Ex. In case Buyer A and Seller B trade 2Y 120 call

Trader	Trade	Hedge	Total
Trader A	+100	-70	+30
Trader B	-100	+70	-30
Market	**0**	**0**	**0**

Example

Ex. In case Buyer C and Seller B trade 2Y 120 call

Trader	Trade	Hedge	Total
Trader A	0	0	0
Trader B	-1	0	-1
Market	**-1**	**0**	**-1**
Client C	+1	0	+1

Example

Ex. In case Buyer C and Seller B trade 2Y 120 call

Trader	Trade	Hedge	Total
Trader A	0	-50	-50
Trader B	-100	+50	-50
Market	**-100**	**0**	**-100**
Client C	+100	0	+100

Problem with Concentration in derivatives

The problem in this case is that **the vega of a derivatives product keeps changing when the market moves!** (without adding on any trades)
Traders need to dynamically hedge their positions.

However, since all the traders are in need of the same kind of products, it is hard to trade fully get what they want, and even if they were able to, hedging cost will be higher.

Example

Ex. In case Buyer C and Seller B trade 2Y 120 call
If the market moves and the vega is doubled...

Trader	Trade	Hedge	Total
Trader A	0	-100	-100
Trader B	-200	+100	-100
Market	**-200**	**0**	**-200**
Client C	+200	0	+200

**Note that the vega is the largest at the strike; if the stock price moves up from the initial price (=100), then the vega increases, ignoring the time decay of the option*

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Heston Model

$$
\begin{cases}
dS = \mu S dt + \sqrt{v} S dW^1 \\
dv = \kappa(\bar{v} - v)dt + \eta\sqrt{v} dW^2 \\
\langle dW^1, dW^2 \rangle = \rho dt.
\end{cases} \tag{1}
$$

with a linear parabolic 2nd order PDE

$$
\begin{aligned}
&\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \kappa\left(\bar{v} - \omega v\right)\frac{\partial V}{\partial v} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} \\
&+ \frac{1}{2} v \eta^2 \frac{\partial^2 V}{\partial v^2} + vS\eta\rho \frac{\partial^2 V}{\partial S \partial v} - rV = 0
\end{aligned} \tag{2}
$$

New Model

We modify the previous Heston model and get similar SDE

$$\begin{cases} dS = \mu S dt + \sqrt{v} S dW^1 \\ dv = \kappa(\bar{v} - v + Q\frac{\partial F}{\partial v})dt + \eta\sqrt{v}dW^2 \\ \langle dW^1, dW^2 \rangle = \rho dt \end{cases} \tag{3}$$

$$\begin{aligned} &\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \kappa\left(\bar{v} - \omega v + Q\frac{\partial F}{\partial v}\right)\frac{\partial V}{\partial v} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} \\ &+ \frac{1}{2}v\eta^2\frac{\partial^2 V}{\partial v^2} + vS\eta\rho\frac{\partial^2 V}{\partial S \partial v} - rV = 0. \end{aligned} \tag{4}$$

$$\begin{aligned} &\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \kappa\left(\bar{v} - \omega v + Q\frac{\partial F}{\partial v}\right)\frac{\partial F}{\partial v} + \frac{1}{2}vS^2\frac{\partial^2 F}{\partial S^2} \\ &+ \frac{1}{2}v\eta^2\frac{\partial^2 F}{\partial v^2} + vS\eta\rho\frac{\partial^2 F}{\partial S \partial v} - rF = 0. \end{aligned} \tag{5}$$

We call this F, the market driver

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Example

We calculate the option price of 2 Year 100% strike call (ATM call) under the assumption that the market driver is a 2 Year 120% strike call (120 call).

**We expect to get more expensive prices from the New Model, since the traders in the OTC market wants to cover the vega they are short from this 120 Call.*

Numerical Result

Risks	Value	Delta	Vega	Vanna	Volga
Heston	2.6058	35.378%	70.940	3.8766	119.001
New Model	3.5121	42.457%	77.188	2.4132	-535.557

Table: Summary for 120 call at $S = 98.255$ and $v = 0.030049$

Risks	Value	Delta	Vega	Vanna	Volga
Heston	11.299	74.117%	79.238	-0.5666	-543.263
New Model	12.116	76.942%	78.824	-1.6201	-961.800

Table: Summary for at-the-money (ATM) call at $S = 98.255$ and $v = 0.030049$

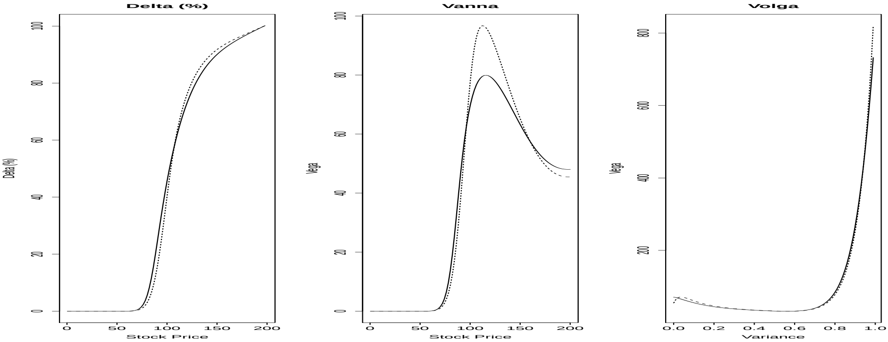
Numerical Result

	120 Call	ATM Call
Heston	17.335%	17.335%
New Model	20.694%	20.090%
Difference	3.359%	2.755%

Table: Implied volatility calculated based on the risk calculated in the Heston model

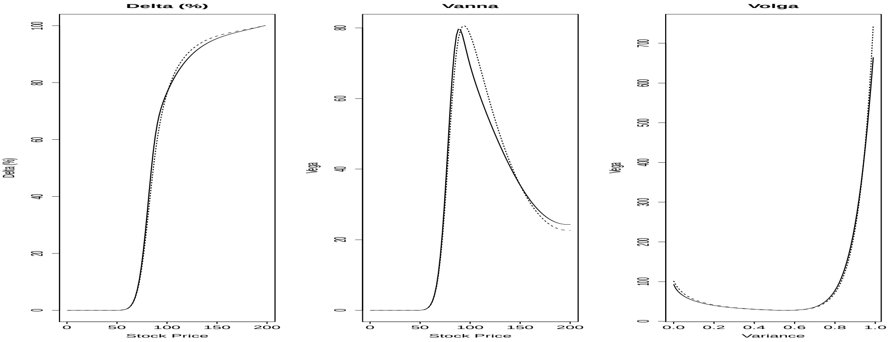
Numerical Result

<u>120 Call</u>



* dotted line → Heston
solid line → New Model

ATM Call



* dotted line → Heston
solid line → New Model

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Approximation

We obtained a pair of PDEs to solve for the market driver F and arbitrary derivative product V, of which one of them was a **semilinear** one.

We now try to find a way to approximate the semilinear PDE with linear PDEs.

Control Problem

Our semilinear PDE can be abstractly written in the following form:

$$u_t + Lu + ru - \kappa Q {u_y}^2 = 0. \tag{6}$$

using L defined by

$$\begin{aligned} -Lu :&= rxu_x + \kappa(v_0 - \alpha y)u_y + \frac{1}{2}x^2 y u_{xx} + \frac{1}{2}\eta^2 y u_{yy} + \eta\rho x y u_{xy} \\ &= a_{ij}u_{ij} + b_i u_i \end{aligned} \tag{7}$$

We convert the semilinear PDE into the one of stochastic control using Legendre transform:

$$\inf_{\pi\in\mathbb{R}} \left(u_t + Lu + ru - \pi u_y + \frac{\pi^2}{4\kappa Q} \right) = 0. \tag{8}$$

PIA

We can calculate the minimizer in this case as

$$\pi = 2\kappa Q u_y \tag{9}$$

We apply the policy improvement algorithm (PIA), which is the iterative algorithm that selects optimal control in each step. The algorithm is as folllows:

1. Take a Markov policy π_0 (where in our case took $\pi_0 \equiv 0$)
2. Solve for V in $L^{\pi_0}V - rV + f^{\pi_0} = 0$ and call the solution V^{π_0}.
3. $\pi_1 = 2\kappa Q(V^{\pi_0})_y$
4. Solve for V in $L^{\pi_1}V - rV + f^{\pi_1} = 0$ and call the solution V^{π_1}.
5. $\pi_2 = 2\kappa Q(V^{\pi_1})_y$
6. $\cdots$

PIA contd.

From the general PIA argument, we can show that this converges.
It is possible that we need to do numbers of iterations in order to get the convergence.
However, from actual numerical calculations, we see that we don't need that many in order to get the convergence. (we show something later that may partly explain this)
The good things about this algorithm are:

1. The semilinear PDE is approximated by a series of linear PDEs.
2. We don't need a lot of iterations in order to get a good approximation of the solution to the semilinear PDE.

Quadratic Local Convergence

We can show that the approximated solution via PIA has Quadratic Local Convergence to the analytic solution in Sobolev norm. In other words, if we define

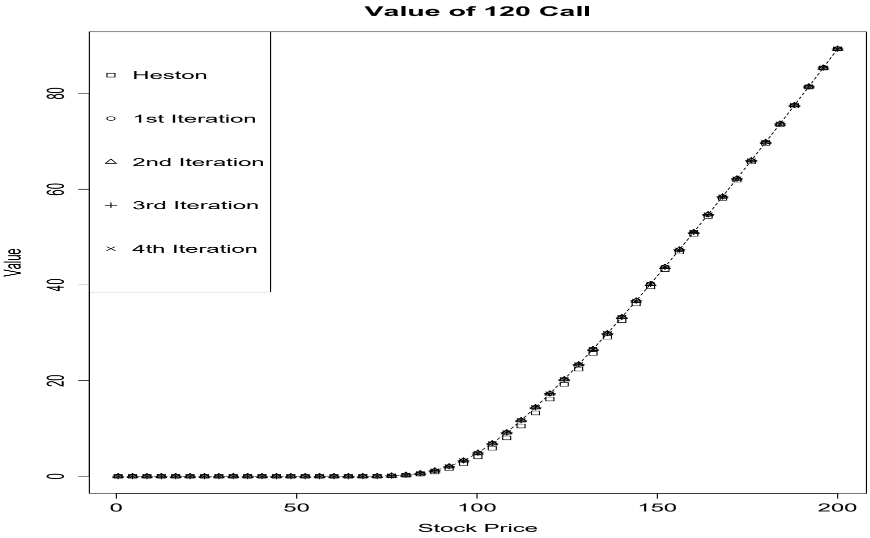
$$W^i = V^{\pi_{i+1}} - V^{\pi_i} \tag{10}$$

then we can show that in case we have Dirichlet boundary condition, we have

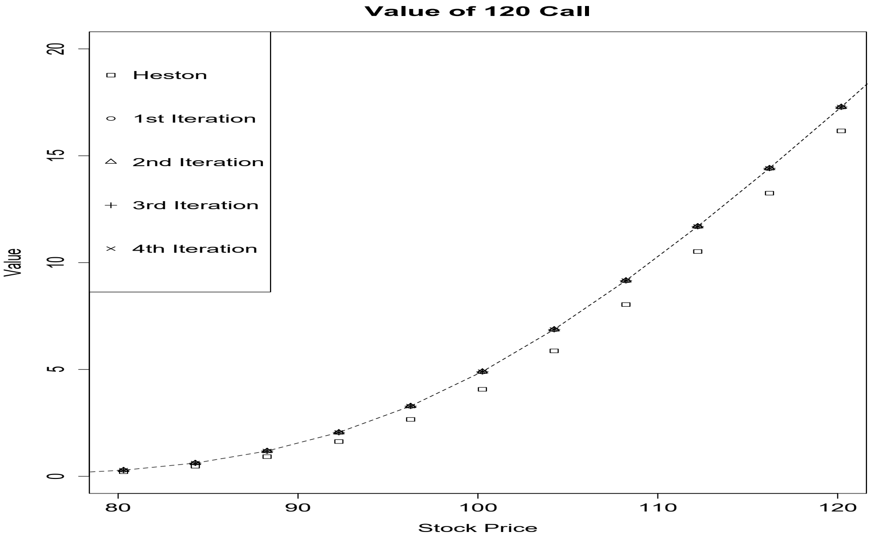
$$\|W^i\|_{2,\beta} \leq C\|W^{i-1}\|_{2,\beta}^2 \tag{11}$$

where C is a constant which depends on the domain, elliptic constant, and the β norm of the coefficients of the differential operator (via use of Schauder's estimate).
This means, if we already had good enough approximation, then the convergence happens quick.

Numerical Result



Numerical Result



Numerical Result

Iteration	0th	1st	2nd	3rd	4th
Difference	2.3177	0.0322	7.72×10^{-6}	0.000	0.000

Table: Largest differences in absolute value between the numerical solutions of the approximated linear PDE and the original semilinear PDE. The figures could be regarded as the differences in percentage against the initial price of the stock as it is set to 100.

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Conclusions

- A new model taking into account the volatility concentration risk from the market driver
- The new model requires solving a second order semilinear parabolic PDE, but can use PIA to efficiently approximate it by a series of linear PDEs

Reference

JM and S. D. Jacka, *A market driver volatility model via policy improvement algorithm*, submitted. [arXiv:1612.00780]

Thank you for your attention