A Structural Model for Default Contagion

Jirô Akahori [1] and Hai-Ha Pham

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Ritsumeikan University, Japan, and Banking University of Hochiminh city, Vietnam

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The Problem

Outline

- Many recent researches have put attention on the systematic risk and contagion since the Asian banking crisis of the late 90s, and the more recent banking crisis of 2007-2008.
- Most of them used directed graphs (network) to model interdependencies.
- The present paper introduces a structural framework— first passage time approach— to model dependent defaults, with a particular interest in their contagion in a very larger market.
- We give an explicit form of contagion probabilities; depth of the contagion, total number of defaults up to a given time, default time of a set of companies, etc

The Model

Let X_t^i denote the firm value process of the i-th company, for $i = 1, 2, \cdots, n$ with $n \geq 2$. Define "default time" by

$$\tau^i := \inf\{s \geq 0 : X_s^i < k^i\},$$

where $k^i \in \mathbf{R}$ is a exogenously given default level for the i-th company.

We assume that $X \equiv (X^1, \cdots, X^n)$ solves the following equation;

$$
\begin{aligned}
X_t^i = x^i - \sum_{j \neq i} C_{i,j} 1_{\{\tau^j < t \wedge \tau^i\}} + \int_0^{t \wedge \tau^i} (\sigma_i(X_s^i) dW^i + \mu^i(X_s^i)\, dt) \\
+ \sum_{j \neq i} \int_{t \wedge \tau^j \wedge \tau^i}^{t \wedge \tau^i} (\sigma_i^j(X_s^i) dW^i + \mu_i^j(X_s)\, dt)
\end{aligned}
\tag{1}
$$

for $i = 1, 2, \cdots, n$, where $W^i, i = 1, \cdots, n$ are independent Brownian motions, and for $i, j = 1, \cdots, n$, $C_{i,j}$ are non-negative constants, σ_i, μ_i, σ_i^j, μ_i^j, each defined on $\mathbb{R}^n$, are smooth function with at most linear growth.

The Model

In a more concise way of saying,

- each component is a diffusion process on each interval from a default time to next one,
- the default of i-th company brings about a jump C_{ij} to j-th company,
- which may causes the default of j-th company.
- The i-th default may also affects the dynamics of the j-th firm value process in terms of its growth rate or the volatility.

Contagion Probabilities

Define the first contagion time by

$$\tau(1) := \min\{\tau_i : i = 1, \cdots, n\}.$$

the j-th contagion time is defined recursively by

$$\tau(j) := \inf\{\tau_k : \tau_k > \tau(j-1)\}, \quad j = 2, 3, \cdots, n,$$

with the convention that $\inf \emptyset = \infty$.

Contagion Probabilities

To price credit derivatives such as CDO or CDS, the distribution of the number of defaulted companies by a fixed time, denoted by N_t, and the joint distribution of $\tau_i, i \in I_0 \subset \{1, \cdots, n\}$ are required.

These are in principle obtained from the joint distribution of

$$(\tau(1), \cdots, \tau(n), D(\tau(1)), \cdots, D(\tau(n))),$$

where

$$D(\tau(k)) = \{i \in \{1, \cdots, n\} : \tau_i = \tau(k)\}, k = 1, \cdots, n,$$

with the convention that $\tau(k) = \infty$ if $D(\tau(k)) = \emptyset$.

Main Results

Key Ideas

The first key idea is that we regard $(\tau(i), X_{\tau(i)})$ as a "renewal-reward" process.

- We shall have a formula of the joint density of $(D(\tau(1)), \tau(1), X_{\tau(1)})$ conditioned by the starting point X_0.
- Here we understand $X_{\tau(1)+t}, t \geq 0$ to be an $R^{D(\tau(1))^c}$-valued process; we are only interested in the survived companies.
- Then, by replacing $\{1, \cdots, n\}$ with $D(\tau(1))^c$ and X_0 with $X_{\tau(1)}$, we obtain the joint distribution of $(D(\tau(2)), \tau(2), X_{\tau(2)})$ conditioned by $X_{\tau(1)}$, thanks to Markov property of X.
- We can repeat this procedure to get the desired joint distribution.

Key Ideas

We can separate the problem of determining the joint distribution of $(D(\tau(1)), \tau(1), X_{\tau(1)})$ into three parts.

- We pretend that we are given the harmonic measure of $X_{\tau(1)-}$ (before the "artificial" jumps): in a simple Brownian case it is known.
- Then the problem reduces to the description of "contagion domain", but it may not be in the form of disjoint union.
- To decompose the domain into disjoint sets, we rely on a recursive equation.

Hierarchal Description

- To take into account that we work on a "renewal" setting described as above, from now on we let the index set of X be arbitrary finite subset.
- In order to specify the initial index set, we put superscript I to the previously defined notations; $\tau^I(1)$, D^I, and so on. We then concentrate on the study of the joint distribution of

$$(D^I(\tau^I(1)), \tau^I(1), X^{I\setminus J}_{\tau^I(1)}). \tag{2}$$

- As we can guess, the joint distribution is obtained from the distribution of $(X^I(\tau^I(1)), \tau^I(1))$.

Contagion Region

The event $\{D^I(\tau^I(1)) = J\}$ is rephrased as the event that $X^I_{\tau^I(1)-}$ hit a set. In fact Let $I := \{i_1, \cdots, i_{\sharp I}\}$ and for a permutation σ over I, or equivalently, $\sigma \in \mathfrak{S}_I$, we put

$$D_{I,\sigma} :=$$

$$\Big\{(x_{i_1}, \cdots, x_{i_{\sharp I}}) \in \mathbf{R}^I : x_{i_{\sigma(1)}} = K^{i_{\sigma(1)}}, x_{i_{\sigma(2)}} \in [K^{i_{\sigma(2)}}, K^{i_{\sigma(2)}} + C_{i_{\sigma(1)}, i_{\sigma(2)}}],$$

$$\cdots, x_{i_{\sigma(\sharp I)}} \in [K^{i_{\sigma(\sharp I)}}, K^{i_{\sigma(\sharp I)}} + \sum_{j=1}^{\sharp I - 1} C_{i_{\sigma(j)}, i_{\sigma(\sharp I)}}]\Big\}.$$

Contagion Region

Then, we have the following

Lemma

For $\emptyset \neq J \subset I$, we have that

$$\begin{aligned} &\{D^I(\tau^I(1)) = J\} \\ &= \left\{ X^I_{\tau^I(1)-} \in \bigcup_{\sigma \in \mathfrak{S}_J} D_{J,\sigma} \times \prod_{i \in I \setminus J} (K^i + \sum_{j \in J} C_{j,i}, \infty) \right\}. \end{aligned}$$

Using this equation, we can get a kind of set inclusion-exclusion formula, by which we can get a disjoint representation of the contagion domain.

“Harmonic Measure”

- So by measuring the domain by the joint distribution of $(X_{\tau(1)-}, \tau(1)-)$ we get what we want.
- If we put an indepndence condition, The distribution is implied by those of X^i to the region $[K^i, \infty)$.

“Harmonic Measure”

Let us be more precise. Let $\tilde{X}$ be a kind of *Business As Usual* process given as

$$\tilde{X}^i_t = x^i + \int_0^t (\sigma_i(\tilde{X}^i_s) dW^i + \mu^i(\tilde{X}^i_s)\, dt),$$

and $\tilde{\tau}_i$ be its default time:

$$\tilde{\tau}_i := \inf\{s > 0 : \tilde{X}^i_s \leq K^i\}.$$

We assume that each of the distribution of $(\tilde{\tau}_i, \tilde{X}^i)$ has a density, and put

$$p_j(s) = \frac{P(\tilde{\tau}_i \in ds)}{ds},$$

and

$$q_j(s, x) = \frac{P(\tilde{\tau}_i > s, \tilde{X}^i \in dx)}{dx}.$$

"Harmonic Measure'

The "harmonic measure", the distribution of of $X^l_{\tau(1)}$ can be obtained by the following

Lemma

For $A \in \mathfrak{B}(G)$,

$$P(X^l_{\tau^l(1)-} \in A, \tau^l(1) \in ds) = \sum_i p_i(s) \int_A \delta_{K^i}(dx_i) \prod_{j \neq i} q_j(s, dx_j), \quad (3)$$

where δ_* is the Dirac delta at $*$.

Let $\emptyset \neq J \subsetneq I$, and define a family of measures

$$h^I(J, s, A) := P(D^I(\tau^I(1)) = J, X^{I\setminus J}_{\tau^I(1)} \in A, \tau^I(1) \in ds)/ds,$$

and

$$g_{J,I}(s, A) := \int_{\prod_{i\in I\setminus J}[K^i,\infty)\times A} \prod_{i\in I\setminus J} q_i(s, x_i + \sum_{j\in J} C_{j,i})\, dx,$$

for $s > 0$ and $A \in \mathfrak{B}(\mathrm{R}^{I\setminus J})$. We also set

$$h^I(s) := P(D^I(\tau^I(1)) = I, \tau^I(1) \in ds), \quad s > 0,$$

and

$$g^{J,I}(s) := \frac{\sum_{j\in J} p_j(s)}{\sum_{i\in I} p_i(s)} g_{I,J}(s, \mathrm{R}^{I\setminus J}).$$

Main Result

The following is our main result.

Theorem

(i) For a finite non-empty $J \subsetneq I$, $s > 0$*and* $A \in \mathfrak{B}(\mathsf{R}^{I \setminus J})$,

$$h^I(J, s, A) = h^I(s) g_{J,I}(s, A). \tag{4}$$

(ii) For $s > 0$,

$$h^I(s) = \left(\sum_{i \in I} p_i(s)\right) \Bigg(1 + \sum_{m=1}^{\sharp I - 1} (-1)^m \sum_{I_m \subsetneq \ldots \subsetneq I_1 \subsetneq I_0} \prod_{l=1}^{m} g^{I_l, I_{l-1}}(s)\Bigg). \tag{5}$$

Thank you! ?????

Discussion