Sensitivity Analysis of Long-term Cash flows

Hyungbin Park
Worcester Polytechnic Institute

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Introduction

Introduction

In finance, we often encounter the quantity of the form:

$$p_T := \mathbb{E}^{\mathbb{P}}[e^{-\int_0^T r(X_t)\,dt} f(X_T)]\,.$$

Purpose: to conduct a **sensitivity analysis** for the quantity p_T with respect to the perturbation of X for large T.

This sensitivity is useful for long-term static investors and for long-dated option prices.

Let $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ be a filtered probability space that has a d-dimensional Brownian motion B with the filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ generated by the B.

A 1

Let $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ be continuous functions and $\xi \in \mathbb{R}^d$. The matrix σ is invertible. Assume that the stochastic differential equation (SDE)

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dB_t\,,\ X_0 = \xi$$

has a unique strong solution X and that the solution is non-explosive.

A 2

The function $r : \mathbb{R}^d \to \mathbb{R}$ is a continuous function.

A 3

The function $f : \mathbb{R}^d \to \mathbb{R}$ is non-negative non-zero measurable.

Given a quadruple of functions (b, σ, r, f) and $\xi \in \mathbb{R}^d$ satisfying A1 - 3, the quantity of interest is

$$p_T = \mathbb{E}_\xi^{\mathbb{P}}[e^{-\int_0^T r(X_t)\,dt} f(X_T)] \,.$$

For the sensitivity w.r.t. the initial value $X_0 = \xi$, we compute

$$\frac{\partial p_T}{\partial \xi}$$

and investigate the behavior of this quantity for large T.

- Why is this an important problem to finance?
  - Hedging market risk
  - How vulnerable to model calibration?

Introduction

Let X_t^ϵ be a perturbed process of X_t

$$dX_t^\epsilon = b_\epsilon(X_t^\epsilon)\,dt + \sigma_\epsilon(X_t^\epsilon)\,dW_t\,,X_0^\epsilon = \xi\ .$$

The perturbed quantity is given by

$$p_T^\epsilon := \mathbb{E}^{\mathbb{P}}[e^{-\int_0^T r(X_s^\epsilon)\,ds} f(X_T^\epsilon)]\ .$$

For the sensitivity analysis, we compute

$$\frac{\partial}{\partial\epsilon}\bigg|_{\epsilon=0} p_T^\epsilon$$

and investigate the behavior of this quantity for large T.

Hansen-Scheinkman decomposition

Fix a quadruple of functions (b, σ, r, f) and $\xi \in \mathbb{R}^d$ satisfying A1 - 3. Consider a pricing operator $\mathcal{P}$ by

$$\mathcal{P}_T f(x) = \mathbb{E}_x^{\mathbb{P}}(e^{-\int_0^T r(X_s)\,ds} f(X_T))$$

so that the expectation is $p_T = \mathcal{P}_T f(\xi)$. For a positive measurable function ϕ and a real number λ such that

$$\mathcal{P}_T \phi(x) = e^{-\lambda T} \phi(x) \quad \text{for } T > 0\,,\ x \in \mathbb{R}^d\,, \tag{2.1}$$

the process

$$M_t^{\phi} := e^{\lambda t - \int_0^t r(X_s)\,ds} \,\frac{\phi(X_t)}{\phi(\xi)}\,,\ 0 \le t \le T$$

is a positive martingale.

Hansen-Scheinkman decomposition

A measure $\mathbb{Q}^{\phi}$ on each $\mathcal{F}_T$ defined by

$$\mathbb{Q}^{\phi}(A) := \mathbb{E}^{\mathbb{P}}_{\xi}(\mathbb{I}_A M^{\phi}_T)$$

for $A \in \mathcal{F}_T$ is called the *eigen-measure* with respect to ϕ.

This definition is well defined on an infinite horizon because

$$\mathbb{E}^{\mathbb{P}}_{\xi}(\mathbb{I}_A M^{\phi}_t) = \mathbb{E}^{\mathbb{P}}_{\xi}(\mathbb{I}_A M^{\phi}_s)$$

for any $A \in \mathcal{F}_s$ and $0 \leq s < t$.

Hansen-Scheinkman decomposition

A 4

There exists a pair (λ, ϕ) of a real number λ and a positive measurable function ϕ satisfying Eq.(2.1) such that the process X is recurrent under the eigen-measure $\mathbb{Q}^\phi$.

In this case, the discount factor $e^{-\int_0^T r(X_t)\,dt}$ can be written as

$$e^{-\int_0^T r(X_s)\,ds} = M_T^\phi e^{-\lambda T} \frac{\phi(\xi)}{\phi(X_T)} \,.$$

This expression is referred to as the *Hansen-Scheinkman decomposition*. We say that (λ, ϕ), λ, ϕ and $\mathbb{Q}^\phi$ are the *recurrent eigenpair*, *recurrent eigenvalue*, *recurrent eigenfunction* and *recurrent eigen-measure*, respectively.

Hansen-Scheinkman decomposition

A recurrent eigenpair may not exist.

The recurrent eigenpair (λ, ϕ) is unique if existent.

Thus, we use notations M and $\mathbb{Q}$ instead of M^{ϕ} and $\mathbb{Q}^{\phi}$, respectively.

(i) Long-Term Risk: An Operator Approach, L. P. Hansen and J.A. Scheinkman, Econometrica, 2009

(ii) Positive Eigenfunctions of Markovian Pricing Operators: Hansen-Scheinkman Factorization, Ross Recovery, and Long-Term Pricing. L. Qin and V. Linetsky, Operations Research, 2016

Hansen-Scheinkman decomposition

A 5

The recurrent eigenfunction ϕ is continuously twice differentiable.

A 6

The process X has an invariant distribution ν under $\mathbb{Q}$.

A 7

The function f is ν-ergodic, that is, f satisfies

$$\mathbb{E}_\xi^{\mathbb{Q}}(f/\phi)(X_T) \to \int (f/\phi)\, d\nu \quad \text{as } T \to \infty\,,$$

and the limit is a finite number.

Hansen-Scheinkman decomposition

In summary, for any given quadruple of functions (b, σ, r, f) and initial value $\xi \in \mathbb{R}^d$ satisfying A1 - 7, we have constructed

$$X,\ \mathcal{P},\ M,\ \mathbb{Q},\ (\lambda, \phi),\ \varphi,\ \nu\,.$$

Then

$$\begin{aligned} p_T = \mathbb{E}^{\mathbb{P}}_{\xi}\big(e^{-\int_0^T r(X_s)\,ds} f(X_T)\big) &= \phi(\xi)\, e^{-\lambda T} \cdot \mathbb{E}^{\mathbb{P}}_{\xi}\big(M_T\,(f/\phi)(X_T)\big) \\ &= \phi(\xi)\, e^{-\lambda T} \cdot \mathbb{E}^{\mathbb{Q}}_{\xi}(f/\phi)(X_T)\,. \end{aligned}$$

Hansen-Scheinkman decomposition

What is it good for?

1) Large-time behavior:

$$\lim_{T\to\infty} \frac{1}{T} \ln p_T = -\lambda \,.$$

2) Dependence on the marginal distributions

Sensitivity analysis

Sensitivity analysis: Delta

The delta: Intuitively,

$$p_T \simeq e^{\lambda T} \phi(\xi)$$

so that

$$\nabla_\xi \ln p_T \simeq \frac{\nabla_\xi \phi}{\phi(\xi)}$$

Observe that

$$\nabla_\xi \ln p_T = \frac{\nabla_\xi p_T}{p_T} = \frac{\nabla_\xi \phi}{\phi(\xi)} + \frac{\nabla_\xi \mathbb{E}_\xi^{\mathbb{Q}}(f/\phi)(X_T)}{\mathbb{E}_\xi^{\mathbb{Q}}(f/\phi)(X_T)} .$$

Need to control: as $T \to \infty$,

$$\nabla_\xi \mathbb{E}_\xi^{\mathbb{Q}}(f/\phi)(X_T)$$

Sensitivity analysis: Delta

Propositioin 3.1

Let (b, σ, r, f) and ξ be a quadruple of functions and an initial value, respectively, satisfying A1 - 7. Assume that the functions $b + \sigma\varphi$ and σ are continuously differentiable with bounded derivatives and that $b + \sigma\varphi$ satisfies the uniform-ellipticity condition. If there exist positive constants $p \geq 2$ and q with $1/p + 1/q = 1$ such that $\mathbb{E}_\xi^{\mathbb{Q}} \|\sigma^{-1}(X_T)Y_T\|^p$ and $\mathbb{E}_\xi^{\mathbb{Q}}(f/\phi)^q(X_T)$ are bounded on $0 \leq T < \infty$, then $\mathbb{E}_\xi^{\mathbb{Q}}(f/\phi)(X_T)$ is continuously differentiable by ξ and $\nabla_\xi \mathbb{E}_\xi^{\mathbb{Q}}(f/\phi)(X_T) \to 0$ as $T \to \infty$.

Here, $Y_t = (Y_{ij,t})_{1\leq i,j\leq d} = (\frac{\partial X_{i,t}}{\partial \xi_j})_{1\leq i,j\leq d}$ is the first variation process

$$dY_{ij,t} = (b + \sigma\varphi)'_i(X_t)Y_{ij,t}\,dt + \sum_{k=1}^{d} \sigma'_{ik}(X_t)Y_{ij,t}\,dB_{k,t}\ ,\ \ Y_0 = I_d$$

and $\|\cdot\|$ the matrix 2-norm.

The rho and vega: Sensitivity with respect to the drift and volatility:

$$dX_t^\epsilon = b_\epsilon(X_t^\epsilon)\, dt + \sigma_\epsilon(X_t^\epsilon)\, dB_t \ , \ X_0^\epsilon = \xi_\epsilon$$

B 1

Let $b_\epsilon(x)$, $\sigma_\epsilon(x)$, $r_\epsilon(x)$, $f_\epsilon(x)$ be functions of variable $(\epsilon, x) \in I \times \mathbb{R}^d$ for a neighborhood I of 0 such that for each x, they are continuously differentiable on $\epsilon \in I$ and $b_0(x) = b(x)$, $\sigma_0(x) = \sigma(x)$, $r_0(x) = r(x)$, $f_0(x) = f(x)$. Let ξ_ϵ be a continuously differentiable function of variable $\epsilon \in I$ and $\xi_0 = \xi$.

Sensitivity analysis

B 2

For each $\epsilon \in I$, the quadruple of functions $(b_\epsilon, \sigma_\epsilon, r_\epsilon, f_\epsilon)$ and initial value ξ_ϵ satisfy A1 - 7.

The meanings of the following objects are self-explanatory:

$$X^\epsilon,\ \mathcal{P}^\epsilon,\ M^\epsilon,\ \mathbb{Q}^\epsilon,\ (\lambda_\epsilon, \phi_\epsilon),\ \varphi_\epsilon,\ \nu_\epsilon\,.$$

We are interested in the perturbed quantity

$$p_T^\epsilon := \mathbb{E}_{\xi_\epsilon}^{\mathbb{P}}(e^{-\int_0^T r_\epsilon(X_s^\epsilon)ds} f_\epsilon(X_T^\epsilon))$$

and the long-term behavior of its sensitivity $\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \ln p_T^\epsilon$.

Sensitivity analysis

Heuristics: when T is large, the term $e^{-\lambda(\epsilon)T}$ dominates the perturbed quantity p_T^ϵ,

$$p_T^\epsilon \simeq e^{-\lambda(\epsilon)T} \phi_\epsilon \epsilon(\xi) \,.$$

We may then expect

$$\left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} p_T^\epsilon \simeq -\lambda'(0)\, T \cdot e^{-\lambda T} \phi(\xi) + e^{-\lambda T} \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} \phi_\epsilon(\xi)$$

Thus,

$$\frac{1}{T} \left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} \ln p_T^\epsilon = \frac{1}{T} \frac{\left.\frac{\partial}{\partial\epsilon}\right|_{\epsilon=0} p_T^\epsilon}{p_T} \simeq -\lambda'(0)$$

Sensitivity analysis

The expectation p_T^ϵ is

$$p_T^\epsilon = e^{-\lambda_\epsilon T} \phi_\epsilon(\xi_\epsilon) \, \mathbb{E}_{\xi_\epsilon}^{\mathbb{Q}_\epsilon}(f_\epsilon/\phi_\epsilon)(X_T^\epsilon) \, .$$

Then

$$\frac{1}{T} \, \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \ln p_T^\epsilon = - \, \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} \lambda_\epsilon + \frac{\frac{\partial}{\partial \epsilon} \, \big|_{\epsilon=0} \, \phi_\epsilon(\xi_\epsilon)}{T \cdot \phi(\xi)}$$

$$+ \, \frac{\frac{\partial}{\partial \epsilon} \, \big|_{\epsilon=0} \, \mathbb{E}_{\xi_\epsilon}^{\mathbb{Q}}(f_\epsilon/\phi_\epsilon)(X_T)}{T \cdot \mathbb{E}_{\xi}^{\mathbb{Q}}(f/\phi)(X_T)} + \frac{\frac{\partial}{\partial \epsilon} \, \big|_{\epsilon=0} \, \mathbb{E}_{\xi}^{\mathbb{Q}_\epsilon}(f/\phi)(X_T^\epsilon)}{T \cdot \mathbb{E}_{\xi}^{\mathbb{Q}}(f/\phi)(X_T)} \, .$$

Need to control:

$$\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \mathbb{E}_{\xi}^{\mathbb{Q}_\epsilon}(f/\phi)(X_T^\epsilon)$$

Sensitivity analysis: Rho

The rho : Let $(b_\epsilon, \sigma, r_\epsilon, f_\epsilon)$ and ξ_ϵ be a quadruple of functions and an initial value, respectively, satisfying B1 - 2.

The perturbed process X^ϵ is

$$dX_t^\epsilon = b_\epsilon(X_t^\epsilon)\,dt + \sigma(X_t^\epsilon)\,dB_t\,,\ X_0^\epsilon = \xi_\epsilon.$$

Define $k_\epsilon := \sigma^{-1} b_\epsilon + \varphi_\epsilon$ and $k := k_0$.

Sensitivity analysis: Rho

Theorem 1

Let $(b_\epsilon, \sigma, r_\epsilon, f_\epsilon)$ *and* ξ_ϵ *be a quadruple of functions and an initial value, respectively, satisfying B1 - 2. Assume that* $\nabla \phi_\epsilon(x)$ *(thus,* $k_\epsilon(x)$*) is continuously differentiable by* ϵ *on* I *for each* x *and that there exists a function* $g : \mathbb{R}^d \to \mathbb{R}$ *such that* $|\frac{\partial k_\epsilon(x)}{\partial \epsilon}| \leq g(x)$ *on* $(\epsilon, x) \in I \times \mathbb{R}^d$*. Suppose that the following conditions hold.*

(i) *There exist positive constants* a, c *and* ϵ_0 *such that for all* $T > 0$

$$\mathbb{E}_\xi^{\mathbb{Q}} e^{\epsilon_0 \int_0^T g^2(X_s)\, ds} \leq c\, e^{aT}\,.$$

There exist positive constants $p \geq 2$ *and* q *with* $1/p + 1/q = 1$ *satisfying the following:*

(ii) *For each* $T > 0$*, there is a positive number* ϵ_1 *such that* $\mathbb{E}_\xi^{\mathbb{Q}} \int_0^T g^{p+\epsilon_1}(X_t)\, dt$ *is finite.*

(iii) $\mathbb{E}_\xi^{\mathbb{Q}} (f/\phi)^q(X_T)$ *is bounded on* $0 \leq T < \infty$*.*

Sensitivity analysis: Rho

Theorem 2 (Continued)

Then, $\mathbb{E}_{\xi}^{\mathbb{Q}_\epsilon}(f/\phi)(X_T^\epsilon)$ is continuously differentiable on $\epsilon \in I$ and

$$\lim_{T\to\infty} \frac{1}{T} \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} \mathbb{E}_{\xi}^{\mathbb{Q}_\epsilon}(f/\phi)(X_T^\epsilon) = 0$$

is obtained.

Sensitivity analysis: Vega

The vega

1st approach: Malliavin calculus with bounded-derivative coefficients. Classical approach for sensitivity analysis

2nd approach: the Lamperti transform for univariate processes. It converts the perturbation of the diffusion term into the drift and the initial value.

Sensitivity analysis: Vega

Conclusion:
Initial value perturbation: eigenfunction determines the zeroth-order growth rate

$$\lim_{T\to\infty} \nabla_\xi \ln p_T = \frac{\nabla_\xi \phi}{\phi(\xi)}$$

Drift and volatility perturbations: eigenvalue determines first-order growth rate

$$\lim_{T\to\infty} \frac{1}{T} \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} \ln p_T^\epsilon = -\lambda'(0)$$

Examples

1) Bond prices
2) Expected utilities

Examples: Bond prices

1) **Bond prices**: The CIR short-interest rate model

Under a risk-neutral measure $\mathbb{P}$, the interest rate r_t follows

$$dr_t = (\theta - ar_t)\,dt + \sigma\sqrt{r_t}\,dB_t\ ,\ 2\theta > \sigma^2.$$

The short-interest rate option price

$$p_T := \mathbb{E}^{\mathbb{P}}[e^{-\int_0^T r_t\,dt} f(r_T)]$$

This is the bond price when $f \equiv 1$.

Want to know the behavior for large T of

$$\frac{\partial p_T}{\partial \theta}\ ,\ \frac{\partial p_T}{\partial a}\ ,\ \frac{\partial p_T}{\partial \sigma}$$

Examples: Bond prices

Assume: $f(r)$ is a nonzero nonnegative continuous function on $r \in [0, \infty)$ with the polynomial growth rate.

The associated second-order equation is

$$\mathcal{L}\phi(r) = \frac{1}{2}\sigma^2 r\phi''(r) + (\theta - ar)\phi'(r) - r\phi(r) = -\lambda\phi(r) \; .$$

The recurrent eigenvalue and its eigenfunction are

$$(\lambda, \phi(r)) := (\theta\kappa, e^{-\kappa r})$$

where $\kappa := \frac{\sqrt{a^2+2\sigma^2}-a}{\sigma^2}$.

For large T, we have that

$$\lim_{T\to\infty} \frac{1}{T} \ln p_T = -\theta\kappa \,,$$

$$\lim_{T\to\infty} \frac{1}{T} \cdot \frac{\partial \ln p_T}{\partial \theta} = -\frac{\sqrt{a^2+2\sigma^2}-a}{\sigma^2} \,,$$

$$\lim_{T\to\infty} \frac{1}{T} \cdot \frac{\partial \ln p_T}{\partial a} = \frac{\theta(\sqrt{a^2+2\sigma^2}-a)}{\sigma^2\sqrt{a^2+2\sigma^2}} \,,$$

$$\lim_{T\to\infty} \frac{1}{T} \cdot \frac{\partial \ln p_T}{\partial \sigma} = \frac{\theta(\sqrt{a^2+2\sigma^2}-a)^2}{\sigma^3\sqrt{a^2+2\sigma^2}} \,,$$

$$\lim_{T\to\infty} \frac{\partial}{\partial r_0} \ln p_T = -\frac{\sqrt{a^2+2\sigma^2}-a}{\sigma^2} \,.$$

Examples: Expected utility

2) **Expected utility**. The Heston model: An asset X_t follows

$$
\begin{aligned}
dX_t &= \mu X_t\, dt + \sqrt{v_t} X_t\, dZ_t \ , \\
dv_t &= (\gamma - \beta v_t)\, dt + \delta \sqrt{v_t}\, dW_t \ ,
\end{aligned}
$$

where Z_t and W_t are two BMs with $\langle Z, W \rangle_t = \rho t$ for $-1 \leq \rho \leq 1$.

Interested in:

$$
\begin{aligned}
p_T :&= \mathbb{E}^{\mathbb{P}}[u(X_T)] = \mathbb{E}^{\mathbb{P}}[X_T^{\alpha}] \\
&= \mathbb{E}^{\mathbb{P}}[e^{\alpha \int_0^T \sqrt{v_t} dZ_t - \frac{\alpha}{2} \int_0^T v_t\, dt}]\, e^{\alpha \mu T} S_0^{\alpha} \\
&= \mathbb{E}^{\hat{\mathbb{P}}}[e^{-\frac{1}{2}\alpha(1-\alpha) \int_0^T v_t\, dt}]\, e^{\alpha \mu T} X_0^{\alpha}
\end{aligned}
$$

where

$$
\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\bigg|_{\mathcal{F}_T} = e^{\alpha \int_0^T \sqrt{v_t} dZ_t - \frac{\alpha^2}{2} \int_0^T v_t\, dt} \ .
$$

Examples: Expected utility

The Heston model

$$\lim_{T\to\infty} \frac{1}{T}\frac{\partial}{\partial\mu} \ln p_T = \alpha$$

$$\lim_{T\to\infty} \frac{1}{T}\frac{\partial}{\partial\gamma} \ln p_T$$

$$= -\frac{1}{2}\alpha(1-\alpha)\cdot\frac{\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}-\beta+\rho\alpha\delta}{\delta^2}$$

$$\lim_{T\to\infty} \frac{1}{T}\frac{\partial}{\partial\beta} \ln p_T = \frac{\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}-\beta+\rho\alpha\delta}{\delta^2\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}}$$

Examples: Expected utility

The Heston model

$$\lim_{T\to\infty} \frac{1}{T}\frac{\partial}{\partial\delta} \ln p_T = -\rho\alpha \cdot \frac{\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}-\beta+\rho\alpha\delta}{\delta^2\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}} + \frac{(\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}-\beta+\rho\alpha\delta)^2}{\delta^3\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}}$$

$$\lim_{T\to\infty} \frac{1}{T}\frac{\partial}{\partial\rho} \ln p_T = -\frac{\alpha\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}-\alpha\beta+\rho\alpha^2\delta}{\delta\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}}$$

$$\lim_{T\to\infty} \frac{\partial}{\partial X_0} \ln p_T = \frac{\alpha}{X_0}$$

$$\lim_{T\to\infty} \frac{\partial}{\partial v_0} \ln p_T = -\frac{1}{2}\alpha(1-\alpha)\cdot\frac{\sqrt{(\beta-\rho\alpha\delta)^2+\delta^2\alpha(1-\alpha)}-\beta+\rho\alpha\delta}{\delta^2}.$$

Thank you !

Related articles

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