Downside risk minimization against a benchmark

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5-th AQFC at Seoul, April 2017

partially based on the joint work with S.J. Sheu

• **Large deviation principle** for the empirical mean of a Bernoulli sequence

$X_1,\ X_2,\ \ldots, X_n, \ldots,$: a Bernoulli sequence,

$$P(X_i = 1) = p, \quad P(X_i = 0) = 1 - p, \quad 0 < p < 1$$

$S_n = \sum_{i=1}^n X_i$,

$$\blacktriangleright \qquad P(\frac{S_n}{n} \le \kappa) \ \sim \ e^{-nI_0(\kappa)}, \qquad \text{as } n \to \infty$$

$$I_0(\kappa) = \sup_{\theta<0}\{\theta\kappa - \rho(\theta)\} : \qquad \text{rate function}$$

$$\rho(\theta) = \frac{1}{n}\log E[e^{\theta S_n}] = \frac{1}{n}\log(1-p+pe^{\theta})^n = \log(1-p+pe^{\theta})$$

Since $\rho(\theta)$ is strictly convex $\rho'(\theta)$ is increasing, and

$$I_0(\kappa) = \infty, \ \text{for } \kappa < \rho'(-\infty) = 0, \qquad I_0(\kappa) = 0, \ \text{for } p = \rho'(0-) \le \kappa$$

• Evaluation of $P(\frac{S_n}{n} \le \kappa)$ is meaningful for $\rho'(-\infty) < \kappa < \rho'(0-)$.

Let us call the interval $(\rho'(-\infty), \rho'(0-))$ "effective domain" of the rate function $I_0(\kappa)$

Note that

$$\rho'(0-) = p = \lim_{n\to\infty} \frac{S_n}{n} \qquad \text{"the law of large numbers"}$$

- "The law of large numbers" rules in the right end point of the "effective domain"

- Large deviation control

Consider minimizing the probability that growth rate of a semi-martingale functional $Y_T(h)$ falls below κ, which may have the following asymptotic behavior in certain circumstances

$$\blacktriangleright \qquad \inf_{h\in\mathcal{H}_{\mathcal{F}}(T)} P(\tfrac{1}{T}Y_T(h) \le \kappa) \sim e^{-TI(\kappa)}, \quad T \to \infty$$

κ: a given target growth rate,

$$Y_T(h) = \int_0^T g(X_s, h_s)ds + \int_0^T \varphi(X_s, h_s)dW_s,$$

X_s: a diffusion process, h_s: a control process,

we are concerned with
- the rate function $I(\kappa)$
- the "effective domain" of κ where $0 < I(\kappa) < \infty$
- (asymptotically) optimal strategies

- The objective of large deviation control is to control arising probabilities of rare events.

- Evaluation of the minimizing probability for κ such that $I(\kappa)$ does not vanish nor diverge is meaningful and we are concerned with the effective domain.

• What is the rate function $I(\kappa)$ in the current case ?

• What rules in the right end point of the "effective domain"?

• The strategies which realize this asymptotics of the minimizing probability are said asymptotically optimal. How to construct them?

- Large deviation control problems provide new aspects in stochastic control theory. Indeed, more probabilistic arguments are required to evaluate such asymptotic behavior of the minimizing probability than the cases of usual control problems.

Usual control problems:

Classical control: $$\inf_{h_\cdot} E[\int_0^T f(X_s, h_s)ds + \Psi(X_T)]$$

Risk-sensitive control: $$\inf_{h_\cdot} E[e^{\theta\{\int_0^T f(X_s,h_s)ds+\Psi(X_T)\}}]$$

Ergodic control: $$\inf_{h_\cdot} \overline{\lim_{T\to\infty}} \frac{1}{T}E[\int_0^T f(X_s, h_s)ds]$$

• **Problem in finance:** "Market model"

Riskless asset:

(0.1) $$dS^0(t) = r(X_t)S^0(t)dt, \quad S^0(0) = s^0.$$

Risky assets:

(0.2) $$\begin{cases} dS^i(t) = S^i(t)\{\alpha^i(X_t)dt + \sum_{k=1}^N \sigma_k^i(X_t)dW_t^k\}, \\ S^i(0) = s^i, \; i = 1, ..., m, \quad N = n + m + 1 \end{cases}$$

Factors:

(0.3) $$\begin{cases} dX_t = \beta(X_t)dt + \lambda(X_t)dW_t, \quad X(0) = x \in R^n \\ \beta : \; R^n \mapsto R^n, \quad \lambda : \; R^n \mapsto R^n \otimes R^N, \end{cases}$$

Benchmark:

(0.4) $$\frac{dL_t}{L_t} = \gamma(X_t)dt + \xi(X_t)^* dW_t, \quad L_0 = l_0, \quad \xi : \; R^n \mapsto R^N$$

Total wealth:

$$V_t = \sum_{i=0}^{m} N_t^i S_t^i$$

N_t^i : Number of the shares

$h_t^i = \frac{N_t^i S_t^i}{V_t}$: Portfolio proportion $\quad i = 0, 1, 2, \ldots, m.$

$h_t := (h_t^1, \ldots, h_t^m)$

Under the self financing condition : $dV_t = \sum_{i=0}^{m} N_t^i dS_t^i$, V_t satisfies

$$\frac{dV_t}{V_t} = \{r(X_t) + h(t)^* \hat{\alpha}(X_t)\}dt + h(t)^* \sigma(X_t) dW_t,$$

and we are concerned with

- $$\inf_{h_\cdot} P(\frac{1}{T} Y_T \le \kappa) := \inf_{h_\cdot} P(\frac{1}{T} \log \frac{V_T(h)}{L_T} \le \kappa) \ \sim \ e^{-TI(\kappa)}, \text{ as } T \to \infty$$

$$\log V_T = \log V_0 + \int_0^T \{-\frac{1}{2} h_s^* \sigma\sigma^*(X_s) h_s + h_s^* \widehat{\alpha}(X_s) + r(X_s)\} dt + \int_0^T h_s^* \sigma(X_s) dW_s,$$

$$\widehat{\alpha}(x) = \alpha(x) - r(x)\mathbf{1}, \qquad \mathbf{1} = (1, 1, \ldots, 1)^*$$

and

$$\log L_T = \log l_0 + \int_0^T \{\gamma(X_s) - \frac{1}{2} \xi^* \xi(X_s)\} ds + \int_0^T \xi(X_s)^* dW_s$$

we have

$$Y_T(h) := \log \frac{V_T}{L_T} = \log \frac{V_0}{l_0} + \int_0^T g(X_s, h_s) ds + \int_0^T \left(h_s^* \sigma(X_s) - \xi^*(X_s)\right) dW_s$$

$$g(x, h) = -\frac{1}{2} h^* \sigma\sigma^*(x) h + h^* \widehat{\alpha}(x) + \widehat{g}_0(x), \qquad \widehat{g}_0(x) = r(x) - \gamma(x) + \frac{1}{2} \xi^* \xi(x)$$

• Previous results (without stochastic benchmark) :

$$(LC) \qquad J_0(\kappa) := \underline{\lim}_{T\to\infty} \frac{1}{T} \inf_{h\in\mathcal{H}(T)} \log P(\frac{1}{T}\log V_T(h) \le \kappa)$$

$$(DR) \qquad J_0(\kappa) = -\sup_{\theta<0}\{\theta\kappa - \chi_0(\theta)\}, \qquad \chi_0'(-\infty) < \kappa < \chi_0'(0-)$$

$$(RS) \qquad \chi_0(\theta) := \underline{\lim}_{T\to\infty} \frac{1}{T} \inf_{h_\cdot} \log E[e^{\theta \log V_T(h)}], \quad \theta < 0.$$

- Duality relationship (DR) between downside risk minimization (LC) and risk-sensitive portfolio optimization over large time (RS) is shown for several models.
- Establishing (DR) and analyzing (RS) give asymp. optimal strategies of (LC):

1. pick up a given constant κ in (LC)
2. take constant $\theta(\kappa)$ which attains the supremum in (DR) with this κ
3. select an asymptotically optimal strategy of (RS) with $\theta = \theta(\kappa)$

c.f.
Hata - N. - Sheu (2010) AAP; N.(2011) QF (Linear Gaussian models)
Hata (2011) APFM (CIR models); N. (2012) AAP (General factor models)
Watanabe (2013) SPA (Hidden Markov models)
N. (2015) BCP vol.105, IMS vol.17,WS; Puhalskii - Stutzer (2017)
Puhalskii (2017); Hata - Sheu (2017); N. - Sheu (2017) Preprints

• New aspects in this talk (I)

We are going to talk about the cases with a stochastic bench mark L_T

$$\inf_{h \in \mathcal{H}_{\mathcal{F}}(T)} P(\frac{1}{T} \log \frac{V_T(h)}{L_T} \leq \kappa) \sim e^{-TI(\kappa)}, \quad T \to \infty.$$

We shall see the duality relationship between (LC) and (RS) over large time holds in a similar context as before. On the other hand, distinct difference appears in the "effective domain" at $\chi'(-\infty)$ although the right endpoint is seen to be ruled by log utility maximization which is a version of "the law of large numbers" .

• Financial meaning of benchmark L_T:

c.f. Davis - Lleo (2015) Risk-sensitive investment management, World Scientific

▶ The concept of the benchmark is ubiquitous in the investment management industry, where 'benchmarks' are preset portfolio or indexes used to compare the performance of an actual portfolio or fund.

▶ The investment committee (for insurance company, pension funds and mutual funds) or the end investor sets the benchmark and mandates the investment manager to produce at least the same level of returns as the benchmark or to outperform it.

▶ What is actually used as a benchmark?

Examples: S&P500 (Standard and Poors 500), MSCI World index,
Barclays Capital Aggregate Bond Index, Nikkei 225, Nomura BPI ...etc.

• New aspects in this talk (II)

We shall also discuss the problems under model uncertainty, where random inputs W_t is not a standard Brownian motion process but includes some distortion nonlinearly depending on the past : (W_t, P^ζ)

$$\log \frac{V_T(h)}{L_T} = \log \frac{v_0}{l_0} + \int_0^T g(X_s, h_s)ds + \int_0^T \{h_s^* \sigma(X_s) - \xi(X_s)^*\} dW_s$$

$$\left.\frac{dP^\zeta}{dP}\right|_{\mathcal{F}_T} = e^{\int_0^T \zeta_s^* dW_s - \frac{1}{2}|\zeta_s|^2 ds}, \quad \widehat{W}_t = W_t - \int_0^t \zeta_s ds \ : \ \text{B.M. under } P^\zeta$$

$$dX_t = \lambda(X_t) d\widehat{W}_t + \{\beta(X_t) + \lambda(X_t)\zeta_t\} dt$$

In this case we formulate the following penalized problem for the distorted model.

$$\blacktriangleright \qquad \inf_h \sup_\zeta P^\zeta\left(\frac{1}{T}\left[\log \frac{V_T(h)}{L_T} + \frac{\mu}{2}\int_0^T |\zeta_s|^2 ds\right] \le \kappa\right) \ \sim \ e^{-T I_\mu(\kappa)}$$

The right endpoint of the effective domain is ruled by "**Robust** log utility maximization" and distinct difference from the "true" models appears in the effective domain $(\chi_\mu'(-\infty), \chi_\mu'(0-))$ at $\theta = -0$ of the rate function $I_\mu(\kappa) = \sup_{\theta<0}\{\theta\kappa - \chi_\mu(\theta)\}$.

1. Problems setting

$\{W_t\}$: N - dim. Brownian motion process on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$

$$\text{State process :} \qquad dX_t = \beta(X_t)dt + \lambda(X_t)dW_t, \quad X_0 = x \in R^n$$

$$\beta : R^n \mapsto R^n, \ \ \lambda : R^n \mapsto R^n \otimes R^N$$

Controlled functional:

$$Y_T(h) = Y_0 + \int_0^T g(X_s, h_s)ds + \int_0^T \{\sigma(X_s)^* h_s - \xi(X_s)\}^* dW_s$$

$$g(x,h) := -\frac{1}{2}h^*\sigma\sigma^*(x)h + h^*\widehat{\alpha}(x) + \widehat{g}_0(x)$$

$$\sigma : R^n \mapsto R^m \otimes R^N, \ \ \widehat{\alpha} : R^n \mapsto R^m, \ \ \widehat{g}_0 : R^n \mapsto R^1 \ \ \xi : R^n \mapsto R^N$$

We are going to study

(1.1) $$J(\kappa) := \underline{\lim}_{T\to\infty} \frac{1}{T} \inf_{h \in \mathcal{H}(T)} \log P(\frac{1}{T}Y_T(h) \le \kappa),$$

$$\mathcal{H}(T) = \{h(t); R^m\text{- valued, progr. m'ble s.t. } P(\int_0^T |h(s)|^2 ds < \infty) = 1\}$$

Consider the risk sensitive control problem:

$$\hat{\chi}(\theta) = \underline{\lim}_{T\to\infty} \frac{1}{T} \inf_{h\in\mathcal{A}(T)} \log E[e^{\theta Y_T(h)}], \quad \theta < 0, \tag{1.2}$$

where the set $\mathcal{A}_T$ of admissible strategies is defined as

$$\mathcal{A}_T = \{h_s \in \mathcal{H}_T; E[e^{M_T^{v,h} - \frac{1}{2}\langle M^{v,h}\rangle_T}] = 1\},$$

$$M_t^{v,h} := \int_0^t [\nabla v(s, X_s)^* \lambda(X_s) + \theta\{h_s^* \sigma(X_s) - \xi(X_s)^*\}] dW_s,$$

where $v(t,x)$ is the solution to the HJB equation for

$$\hat{v}(t,x) + \theta y_0 = \inf_{h\in\mathcal{A}(T)} \log E[e^{\theta Y_T(h)}]$$

• We establish the duality relationship between (1.1) and (1.2), and construct asymptotically optimal strategies. We then examine the effective domain of $I(\kappa)$ and, in particular, see what rules in the right end point of the "effective domain".

2. Duality theorem

$$Y_t(h) = y_0 + \int_0^t g(X_s, h_s)ds + \int_0^t \{h_s^* \sigma(X_s) - \xi(X_s)^*\} dW_s.$$

Thus,

$$(2.1)\qquad \begin{aligned} e^{\theta Y_T(h)} &= e^{\theta y_0} e^{\theta \int_0^T g(X_s,h_s)ds + \theta \int_0^T \{h_s^* \sigma(X_s) - \xi(X_s)^*\} dW_s} \\ &= e^{\theta y_0} e^{\theta \int_0^T \eta(X_s,h_s)ds + \theta M_T^h - \frac{\theta^2}{2}\langle M^h \rangle_T}, \end{aligned}$$

where

$$M_t^h = \int_0^t \{h_s^* \sigma(X_s) - \xi(X_s)^*\} dW_s, \qquad \eta(x,h) = g(x,h) + \frac{\theta}{2}|\sigma(x)^* h - \xi(x)|^2.$$

If $E[e^{\theta M_T^h - \frac{\theta^2}{2}\langle M^h \rangle_T}] = 1$, then the value function can be written as

$$(2.2)\qquad \hat{J}(t,x) = \theta y_0 + \inf_{h_\cdot \in \mathcal{A}(T)} \log E^h[e^{\theta \int_0^{T-t} \eta(X_s,h_s)ds}] \equiv \theta y_0 + \hat{v}(t,x)$$

since

$$P^h(A) = E[e^{\theta M_T^h - \frac{\theta^2}{2}\langle M^h\rangle_T} : A]$$

defines a probability measure and

$$W_t^h := W_t - \theta\int_0^t \{\sigma^*(X_s)h_s - \xi(X_s)\}ds$$

is a Brownian motion process under P^h and

SDE:

$$dX_t = \{\beta(X_t) + \theta\lambda(X_t)(\sigma^*(X_t)h_t - \xi(X_t))\}dt + \lambda(X_t)dW_t^h, \quad X_0 = x \tag{2.3}$$

is regarded as the controlled dynamics.

The HJB equation for $\hat{v}(t,x)$ is seen to be

$$\begin{cases} \dfrac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 v] + \frac{1}{2}(Dv)^*\lambda\lambda^* Dv \\ \qquad\qquad + \inf_h\{[\beta + \theta\lambda(\sigma^* h - \xi)]^* Dv + \theta\eta(x,h)\} = 0, \\ v(T,x) = 0, \end{cases} \tag{2.4}$$

or,

$$(2.4)' \qquad \begin{cases} \dfrac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 v] + \beta_\theta^* Dv + \frac{1}{2}(Dv)^*\lambda N_\theta^{-1}\lambda^* Dv + U_\theta = 0, \\ v(T,x) = 0, \end{cases}$$

where

$$\beta_\theta = \beta + \theta\lambda N_\theta^{-1}(\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha} - \xi), \qquad N_\theta^{-1} = I + \tfrac{\theta}{1-\theta}\sigma^*(\sigma\sigma^*)^{-1}\sigma$$

$$U_\theta = \frac{\theta^2}{2}\{\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha} - \xi\}^* N_\theta^{-1}\{\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha} - \xi\} + \theta(\frac{1}{2}\widehat{\alpha}^*(\sigma\sigma^*)^{-1}\widehat{\alpha} + \widehat{g}_0(x))$$

► $$\frac{1}{1-\theta}I \le N_\theta^{-1} \le I$$

$$(2.4)' \qquad \begin{cases} \dfrac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 v] + \beta^* Dv \\ \quad + \frac{1}{2}\{\lambda^* Dv + \theta(\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha} - \xi)\} N_\theta^{-1}\{\lambda^* Dv + \theta(\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha} - \xi)\} \\ \qquad + \theta(\dfrac{1}{2}\widehat{\alpha}^*(\sigma\sigma^*)^{-1}\widehat{\alpha} + \widehat{g}_0(x)) = 0 \\ v(T,x) = 0, \end{cases}$$

Assumptions

(2.5) $$\lambda,\ \beta,\ \sigma,\ \widehat{\alpha},\ \widehat{g}_0,\ \text{ and }\ \xi \text{ are globally Lipschitz and smooth},$$

(2.6) $$\widehat{g}_0 \text{ is bounded below and } \xi \text{ is bounded},$$

(2.7) $$\begin{cases} c_1|q|^2 \le q^*\lambda\lambda^*(x)q \le c_2|q|^2, \quad c_1,\ c_2 > 0, \quad q \in R^n, \\ c_1|\zeta|^2 \le \zeta^*\sigma\sigma^*(x)\zeta \le c_2|\zeta|^2, \quad \zeta \in R^m. \end{cases}$$

Proposition 1 *Under assumptions (2.5) - (2.7), H-J-B equation (2.4) has a solution such that*

$$v(t,x) \le K_0,$$

$$v \in C^{1+\delta/2,2+\delta}([0,T) \times R^n),$$

$$\frac{\partial v}{\partial t} \ge -C, \qquad C > 0$$

$$|Dv|^2 + k(\frac{\partial v}{\partial t} + C) \le C'(|\nabla N_\theta^{-1}|_{2r}^2 + |N_\theta^{-1}|_{2r}^2 + |\nabla(\lambda\lambda^*)|_{2r}^2 + |\nabla\beta_\theta|_{2r}$$
$$+|\beta_\theta|_{2r}^2 + |U_\theta| + |\nabla U_\theta| + 1), \qquad x \in B_r,\ \ t \in [0,T),$$

where $k > 0$ is a positive constant, C' a positive constant depending on c_1, c_2, C, θ, k and n but not on r, and $-C$ is the lower bound of $-U_\theta$.

Remarks.

• c.f. N. '96 SICON, Bensoussan-Frehse-N. '98 AMO, N. '04 SICON, Hata-N.-Sheu '17 Preprint, for the proof of Proposition 1.

• Note that the solution v to HJB equation (2.4) such that $|v(t,x)| \leq M(1+|x|^2)$, $M > 0$ is unique (c.f. F. Da Lio and O. Ley '06 SICON) and the set $\mathcal{A}_T$ of admissible strategies is defined by the solution specified in such a way.

We can see that the infimum in (2.4) is attained by

$$\widehat{h}(t,x) := \frac{(\sigma\sigma^*)^{-1}}{1-\theta}\{\sigma\lambda^*\nabla v(t,x) + \widehat{\alpha}(x) - \theta\sigma\xi(x)\}. \tag{2.8}$$

and the optimal strategy is constructed from this function. Indeed, we have the following **verification theorem.**

Proposition 2 *Let $v(t,x;T)$ be a solution to H-J-B equation (2.4). Then, under assumptions (2.5)-(2.7), the following verification holds*

$$\inf_{h\in\mathcal{A}_T} \log E[e^{\theta Y_T(h)}] = \log E[e^{\theta Y_T(\widehat{h})}] = v(0,x;T) + \theta y_0,$$

where, $\widehat{h}_t = \widehat{h}(t,X_t)$, and the set $\mathcal{A}_T$ of admissible strategies is defined as

$$\mathcal{A}_T = \{h_s \in \mathcal{H}_T; E[e^{M_T^{v,h} - \frac{1}{2}\langle M^{v,h}\rangle_T}] = 1\},$$

$$M_t^{v,h} := \int_0^t [\nabla v(s,X_s)^*\lambda(X_s) + \theta\{h_s^*\sigma(X_s) - \xi(X_s)^*\}]dW_s,$$

Remark. c.f. Davis-Lleo '08 QF, concerning benchmarked risk-sensitive asset management for **linear Gaussian models** in the case of $Y_T(h) = \log \frac{V_T(h)}{L_T}$, and also their WS volume (2015),
c.f. N. '15 in WS volume in the case without benchmark for general models

Ergodic type HJB equation

(2.9) $$\chi(\theta) = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\overline{v}] + \frac{1}{2}(D\overline{v})^*\lambda\lambda^* D\overline{v} + \inf_h \{[\beta + \theta\lambda(\sigma^* h - \xi)]^* D\overline{v} + \theta\eta(x,h)\},$$

(2.9)′ $$\chi(\theta) = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\overline{v}] + \beta_\theta^* D\overline{v} + \frac{1}{2}(D\overline{v})^*\lambda N_\theta^{-1}\lambda^* D\overline{v} + U_\theta.$$

Proposition 3 *(i) Under the condition that*

(2.10) $$\widehat{\alpha}^*(\sigma\sigma^*)^{-1}\widehat{\alpha} \to \infty, \quad |x| \to \infty$$

there exists a solution $(\chi(\theta), \overline{v})$ to (2.9) such that $\overline{v}$ is bounded above. Further, such a solution is unique up to additive constants with respect to $\overline{v}$.

(ii) We further assume that there exists a positive constant $c > 0$ such that

(2.11) $$\varlimsup_{|x|\to\infty} \frac{H(x,cx)}{1+|x|^2} < 0, \quad H(x,p) := \beta_\theta^* p + \frac{1}{2}p^*\lambda N_\theta^{-1}\lambda^* p + U_\theta.$$

Then,

$$v(0,x;T) - (\overline{v}(x) + \chi(\theta)T) \to c_\infty, \quad T \to \infty,$$

uniformly on each compact set, where c_∞ is a constant.

(iii) If we assume that

$$(2.11)' \qquad \hat{\alpha}^*(\sigma\sigma^*)^{-1}\hat{\alpha} \geq c_\alpha|x|^2 - c'_\alpha \quad \exists c_\alpha, c'_\alpha > 0$$

then, (2.11) holds and moreover we have

$$\bar{v}(x) \leq -c_v|x|^2 + c'_v, \quad \exists c_v,\ c'_v > 0$$

Remarks.

- cf. Bensoussan-Frehse '92; N. '96 SICON, '12 AAP, as for (i)
- cf. Ichihara-Sheu '13 SIMA, and N. '15 in WS volume, for (ii)
- cf. N. '12 AAP for (iii)

- Further, we have the following estimates for the solution $\bar{v}$ of EHJB

$$-\underline{c}_v|x|^2 - \underline{c}'_v \leq \bar{v}(x) \leq -c_v|x|^2 + c'_v, \quad \exists c_v,\ c'_v > 0,\ \ \underline{c}_v,\ \underline{c}'_v > 0$$

since the gradient estimates

$$|\nabla\bar{v}|^2(x) \leq C(1 + |x|^2), \quad C > 0$$

holds.

Convexity Rewrite (2.4)' as follows

$$\frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 v] + \beta^*\nabla v$$
$$+\frac{1}{2}\{\lambda^*\nabla v + \theta(\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha} - \xi)\}^* N_\theta^{-1}\{\lambda^*\nabla v + \theta(\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha} - \xi)\}$$
$$+\frac{\theta}{2}\widehat{\alpha}^*(\sigma\sigma^*)^{-1}\widehat{\alpha} + \theta\widehat{g}_0(x) = 0,$$
$$v(T,x) = 0$$

It is regarded as the HJB equation

$$\frac{\partial v}{\partial t} + \tfrac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 v] + \beta^*\nabla v + \sup_{z\in R^N}[-\frac{1}{2}z^* N_\theta z + z^*\{\lambda^*\nabla v + \theta(\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha} - \xi)\}]$$
$$+\frac{\theta}{2}\widehat{\alpha}^*(\sigma\sigma^*)^{-1}\widehat{\alpha} + \theta\widehat{g}_0(x) = 0$$
$$v(T,x) = 0$$

It corresponds to the stochastic control problem having the value function

$$v(t,x;T) = \sup_{Z_\cdot \in \mathcal{Z}_T} E[\int_0^{T-t} \Phi(Y_s, Z_s) ds]$$

with

$$\Phi(x,z) = -\frac{1}{2} z^* N_\theta z + \theta z^* (\sigma^* (\sigma\sigma^*)^{-1} \widehat{\alpha} - \xi) + \frac{\theta}{2} \widehat{\alpha}^* (\sigma\sigma^*)^{-1} \widehat{\alpha} + \theta \widehat{g}_0(x)$$

and the control dynamics Y_t governed by

(2.15) $$dY_t = \lambda(Y_t) dW_t + \{\beta(Y_t) + \lambda(Y_t) Z_t\} dt, \quad Y_0 = x$$

$$\mathcal{Z}_T = \{z_t; z_t \text{ is progr. m'ble s.t. } E[\int_0^T |Z_s|^2 ds] < \infty\}$$

Lemma 1 *The solution of HJB equation (2.4)' is convex with respect to θ.*

Indeed, since $\Phi(x,z)$ is a linear function of θ, $\widehat{v}(t,x;T)$ is convex . Thus, solution $v(t,x)$ to the HJB equation turns out to be convex.

- $\chi(\theta) = \lim_{T\to\infty} \frac{1}{T} v(0,x;T)$ is also convex. (cf. Proprosition 3 (ii))

- Solution $(\chi(\theta), \bar{v})$ to EHJB and its derivative

  - $$\chi(\theta) = \frac{1}{2}\mathrm{tr}[\lambda\lambda^*(x)D^2\bar{v}] + \frac{1}{2}(D\bar{v})^*\lambda\lambda^*(x)D\bar{v} + \inf_{h\in R^m}[\{\beta(x) + \theta\lambda(x)(\sigma^*(x)h - \xi(x))\}^* D\bar{v} + \theta\eta(x,h)],$$

  - $$\chi(\theta) = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\bar{v}] + \beta_\theta^* D\bar{v} + \frac{1}{2}(D\bar{v})^*\lambda N_\theta^{-1}\lambda^* D\bar{v} + U_\theta$$

$$\chi(\theta) = \overline{L}\bar{v} - \frac{1}{2}(D\bar{v})^*\lambda N_\theta^{-1}\lambda^* D\bar{v} + U_\theta$$

$$\overline{L}\phi := \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\phi] + \beta_\theta^* D\phi + (D\bar{v})^*\lambda N_\theta^{-1}\lambda^* D\phi \ : \ \text{ergodic}$$

Equation of the formal derivatives of EHJB:

$$(*) \qquad \chi'(\theta) = \overline{L}u + (\frac{\partial\beta_\theta}{\partial\theta})^* D\bar{v} + \frac{1}{2}(D\bar{v})^*\lambda\frac{\partial N_\theta^{-1}}{\partial\theta}\lambda^* D\bar{v} + \frac{\partial U_\theta}{\partial\theta}$$

$$
(P) \qquad \left\{ \begin{array}{c} \rho(\theta) = \overline{L}u + (\frac{\partial \beta_\theta}{\partial \theta})^* D\bar{v} + \frac{1}{2}(D\bar{v})^* \lambda \frac{\partial N_\theta^{-1}}{\partial \theta} \lambda^* D\bar{v} + \frac{\partial U_\theta}{\partial \theta} \\ \\ u \in W^{2,p}_{loc}, \quad \sup_{B_R^c} \frac{|u(x)|}{|\bar{v}(x)|} < \infty, \quad \exists R > 0, \end{array} \right.
$$

under

$$
\sup_{B_R^c} \frac{|f_\theta(x)|}{\overline{L}\ \bar{v}(x)} < \infty, \quad f_\theta(x) := (\frac{\partial \beta_\theta}{\partial \theta})^* D\bar{v} + \frac{1}{2}(D\bar{v})^* \lambda \frac{\partial N_\theta^{-1}}{\partial \theta} \lambda^* D\bar{v} + \frac{\partial U_\theta}{\partial \theta}
$$

Theorem 1 *Let us assume (2.5)- (2.7) and (2.11)'. Then, we have a solution* $(\rho(\theta), u)$ *to (P) such that* $u \in W^{2,p}_{loc}$, $\sup_{B_R^c} \frac{|u(x)|}{|\bar{v}(x)|} < \infty$, *if and only if*

$$
\int \{f_\theta(x) - \rho(\theta)\} m_\theta(dx) = 0,
$$

where m_θ *is an invariant measure of* $\overline{L}$. *Further, this solution* u *is unique up to additive constants and* $\rho(\theta) = \chi'(\theta)$.

Theorem 2 *i) Assume (2.5)-(2.7) and (2.11)'. Then, for* κ *such that* $\chi'(-\infty) < \kappa < \chi'(0-)$*, we have*

$$\varliminf_{T\to\infty} \frac{1}{T} \inf_{h_\cdot} \log P(\frac{1}{T} Y_T(h) \le \kappa) = -\sup_{\theta<0}\{\theta\kappa - \chi(\theta)\}.$$

Further, take $\theta(\kappa)$ *which attains the supremum in the right hand side and set* $\overline{h}_t^{(T)} = \hat{h}(X_t, \nabla v(t, X_t))$ *with*

$$\hat{h}(x,p) = \frac{(\sigma\sigma^*)^{-1}}{1-\theta}\{\sigma\lambda^*\nabla v(t,x) + \hat{\alpha}(x) - \theta\sigma\xi(x)\},$$

where $\theta = \theta(\kappa)$ *and* $v(t,x)$ *is the solution to the HJB equation (2.4) with* $\theta = \theta(\kappa)$*.*

Then,

$$\lim_{T\to\infty} \frac{1}{T} \log P(\frac{1}{T} Y_T(\overline{h}^{(T)}) \le \kappa) = -\sup_{\theta<0}\{\theta\kappa - \chi(\theta)\}$$

ii) If the solution $\bar{v}$ *to equation (2.9) with* $\theta = \theta(\kappa)$ *satisfies*

$$(\nabla\bar{v})^*\lambda\sigma^*(\sigma\sigma^*)^{-1}\sigma\lambda^*\nabla\bar{v} - (\widehat{\alpha} - \theta\sigma\xi)^*(\sigma\sigma^*)^{-1}(\widehat{\alpha} - \theta\sigma\xi) \to -\infty, \quad |x| \to \infty,$$

then, by setting $\tilde{h}_t = \tilde{h}(X_t, \nabla\bar{v}(X_t))$ *with*

$$\tilde{h}(x,p) = \frac{1}{1-\theta}(\sigma\sigma^*)^{-1}\{\sigma\lambda^*(x)p + \widehat{\alpha}(x) - \theta\sigma\xi(x)\},$$

we have

$$\begin{aligned}\lim_{T\to\infty} \tfrac{1}{T}\log P(\tfrac{1}{T}Y_T(\tilde{h}) \le \kappa) &= \underline{\lim}_{T\to\infty} \inf_{h_\cdot} \tfrac{1}{T}\log P(\tfrac{1}{T}Y_T(h) \le \kappa) \\ &= \inf_{h_\cdot} \underline{\lim}_{T\to\infty} \tfrac{1}{T}\log P(\tfrac{1}{T}Y_T(h) \le \kappa) \\ &= -\sup_{\theta<0}\{\theta\kappa - \chi(\theta)\}.\end{aligned}$$

Remark. Similar results are obtained by Puhalskii (Preprint in March 2017) by using Kantorovich-Rubinstein distance under the assumptions that

$$\beta(x)^*x \le -c|x|^2 + c', \qquad y^*\lambda N_{-\infty}^{-1}\lambda^* y \ge c_0|y|^2$$

$$\xi^* N_{-\infty}^{-1}\{I - \lambda^*(\lambda N_{-\infty}^{-1}\lambda^*)^{-1}\lambda\}N_{-\infty}^{-1}\xi > c'' > 0$$

$$N_{-\infty}^{-1} = I - \sigma^*(\sigma\sigma^*)^{-1}\sigma$$

3. Characterization of $\chi'(0-)$ by "the law of large numbers"

$$(A) \qquad \beta(x)^* x \leq -c_\beta |x|^2 + c'_\beta, \quad c_\beta,\ c'_\beta > 0$$

Proposition 4 *Besides of the assumptions of Theorem 1 we assume (A). Then,*

$$\begin{aligned}\chi'(0-) &= \lim_{T\to 0} \frac{1}{T} E[\int_0^T \{\frac{1}{2}\widehat{\alpha}^*(\sigma\sigma^*)^{-1}\widehat{\alpha} + \widehat{g}_0\}(X_s)ds] \\ &= \int \{\frac{1}{2}\widehat{\alpha}^*(\sigma\sigma^*)^{-1}\widehat{\alpha} + \widehat{g}_0\}(x)m(dx) \\ &= \sup_{h_\cdot} \underline{\lim}_{T\to 0} \frac{1}{T} E[Y_T(h)].\end{aligned}$$

and $\quad I(\kappa) := \sup_{\theta<0}\{\theta\kappa - \chi(\theta)\} = 0, \quad$ *for* $\ \kappa \geq \chi'(0-)$

Remark. It is nothing but **log utility maximization** when $\quad Y_T(h) = \log \frac{V_T(h)}{L_T}$

4. Linear Gaussian case

$$\sigma(x) = \Sigma, \qquad \beta(x) = Bx + b, \qquad \lambda(x) = \Lambda,$$

$$\xi(x) = \Xi, \qquad \widehat{\alpha}(x) = \widehat{A}^* x + \widehat{a}, \qquad \widehat{g}_0(x) = g_1^* x + g_0$$

Explicit representation: ▶ $v(t,x) = \frac{1}{2} x^* P(t) x + q(t)^* x + l(t),$

- $$\dot{P} + K_1^* P + P K_1 + P \Lambda N_\theta^{-1} \Lambda^* P + \frac{\theta}{1-\theta} \widehat{A}^* (\Sigma\Sigma^*)^{-1} \widehat{A} = 0, \quad P(T) = 0$$

$$K_1 := B + \theta \Lambda N_\theta^{-1} \Sigma^* (\Sigma\Sigma^*)^{-1} \widehat{A}$$

- $$\dot{q}(t) + (K_1 + \Lambda N_\theta^{-1} \Lambda^* P(t))^* q(t) + P(t)\{b + \theta \Lambda N_\theta^{-1} (\Sigma^* (\Sigma\Sigma^*)^{-1} \widehat{a} - \Xi)\}$$

$$\frac{\theta}{1-\theta} \widehat{A}^* (\Sigma\Sigma^*)^{-1} (\widehat{a} - \theta \Sigma \Xi) + \theta g_1 = 0, \quad q(T) = 0$$

- $$\dot{l} + \frac{1}{2} \mathrm{tr}[\Lambda\Lambda^* P] + \{b + \theta \Lambda N_\theta^{-1} (\Sigma^* (\Sigma\Sigma^*)^{-1} \widehat{a} - \Xi)\} q + \frac{1}{2} q^* \Lambda N_\theta^{-1} \Lambda^* q$$

$$+ \frac{\theta^2}{2} \{\Sigma^* (\Sigma\Sigma^*)^{-1} \widehat{a} - \Xi\}^* N_\theta^{-1} \{\Sigma^* (\Sigma\Sigma^*)^{-1} \widehat{a} - \Xi\} + \frac{\theta}{2} \widehat{a} (\Sigma\Sigma^*)^{-1} \widehat{a} + \theta g_0 = 0,$$

$$l(T;T) = 0$$

Explicit representation for a stationary solution: ► $\bar{v}(x) = \frac{1}{2}x^*\bar{P}x + \bar{q}^*x$

- $$K_1^*\bar{P} + \bar{P}K_1 + \bar{P}\Lambda N_\theta^{-1}\Lambda^*\bar{P} + \frac{\theta}{1-\theta}\widehat{A}^*(\Sigma\Sigma^*)^{-1}\widehat{A} = 0$$

- $$(K_1 + \Lambda N_\theta^{-1}\Lambda^*\bar{P})^*\bar{q} + \bar{P}\{b + \theta\Lambda N_\theta^{-1}(\Sigma^*(\Sigma\Sigma^*)^{-1}\hat{a} - \Xi)\}$$
$$+\frac{\theta}{1-\theta}\widehat{A}^*(\Sigma\Sigma^*)^{-1}(\hat{a} - \theta\Sigma\Xi) + \theta g_1 = 0.$$

- $$\chi(\theta) = \frac{1}{2}\mathrm{tr}[\Lambda\Lambda^*\bar{P}] + \{b + \theta\Lambda N_\theta^{-1}(\Sigma^*(\Sigma\Sigma^*)^{-1}\hat{a} - \Xi)\}\bar{q} + \frac{1}{2}\bar{q}^*\Lambda N_\theta^{-1}\Lambda^*\bar{q}$$
$$+\frac{\theta^2}{2}\{\Sigma^*(\Sigma\Sigma^*)^{-1}\hat{a} - \Xi\}^* N_\theta^{-1}\{\Sigma^*(\Sigma\Sigma^*)^{-1}\hat{a} - \Xi\} + \frac{\theta}{2}\hat{a}(\Sigma\Sigma^*)^{-1}\hat{a} + \theta g_0$$

- Under one of the following conditions:

(G) $G := B - \Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}\widehat{A}$ is stable

(B) B is stable

(ΛA) $\Lambda\Lambda^*$, and $\widehat{A}^*\widehat{A}$ are positive definite matrices

we have the solution $\bar{P}$ such that

$$\lim_{T\to\infty} P(t;T) = \bar{P}, \quad \text{and} \quad K_1 + \Lambda N_\theta^{-1}\Lambda^*\bar{P} \quad \text{is stable.}$$

Further, we have

$$\lim_{T\to\infty} q(t;T) = \bar{q}, \qquad \text{and} \qquad \lim_{T\to\infty} \frac{l(0;T)}{T} = \chi(\theta),$$

Derivatives:

$\bar{P}$ is differentiable and $\frac{d\bar{P}}{d\theta}$ satisfies

$$(K_1 + \Lambda N_\theta^{-1}\Lambda\bar{P})^*\tfrac{d\bar{P}}{d\theta} + \tfrac{d\bar{P}}{d\theta}(K_1 + \Lambda N_\theta^{-1}\Lambda\bar{P})$$

$$+\tfrac{1}{(1-\theta)^2}(\Sigma\Lambda^*\bar{P} + \widehat{A})^*(\Sigma\Sigma^*)^{-1}(\Sigma\Lambda^*\bar{P} + \widehat{A}) = 0$$

$\bar{q}$ is differentiable and

$$(K_1 + \Lambda N_\theta^{-1}\Lambda\bar{P})^*\frac{d\bar{q}}{d\theta} + (\tfrac{dK_1}{d\theta} + \Lambda\frac{dN_\theta^{-1}}{d\theta}\Lambda\bar{P} + \Lambda N_\theta^{-1}\Lambda^*\frac{d\bar{P}}{d\theta})^*\bar{q}$$

$$+\frac{d\bar{P}}{d\theta}\{b + \theta\Lambda N_\theta^{-1}(\Sigma^*(\Sigma\Sigma^*)^{-1}\widehat{a} - \Xi)\}$$

$$+\bar{P}\{\tfrac{1}{(1-\theta)^2}\Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}\widehat{a} - \Lambda N_\theta^{-1}N_\theta^{-1}\Xi\}$$

$$+\frac{1}{(1-\theta)^2}\widehat{A}^*(\Sigma\Sigma^*)^{-1}(\widehat{a} - \theta\Sigma\Xi) - \frac{\theta}{1-\theta}\widehat{A}^*(\Sigma\Xi)^*(\Sigma\Sigma^*)^{-1}\Sigma\Xi + g_1 = 0$$

Then, we have

$$\chi'(\theta) = \frac{1}{2}\mathrm{tr}[\Lambda\Lambda^*\frac{\partial \bar{P}}{\partial\theta}] + \{b + \theta\Lambda N_\theta^{-1}(\Sigma^*(\Sigma\Sigma^*)^{-1}\widehat{a} - \Xi)\}^*\frac{\partial\bar{q}}{\partial\theta}$$

$$+\{\Lambda N_\theta^{-1}(\Sigma^*(\Sigma\Sigma^*)^{-1}\widehat{a} - \Xi) + \theta\Lambda\frac{\partial N_\theta^{-1}}{\partial\theta}(\Sigma^*(\Sigma\Sigma^*)^{-1}\widehat{a} - \Xi)\}^*\bar{q}$$

$$+\frac{\partial\bar{q}^*}{\partial\theta}\Lambda N_\theta^{-1}\Lambda^*\bar{q} + \frac{1}{2}\bar{q}^*\Lambda\frac{\partial N_\theta^{-1}}{\partial\theta}\Lambda^*\bar{q} + \frac{1}{2}\widehat{a}^*(\Sigma\Sigma^*)^{-1}\widehat{a} + g_0$$

$$+\theta\{\Sigma^*(\Sigma\Sigma^*)^{-1}\widehat{a} - \Xi\}^*N_\theta^{-1}\{\Sigma^*(\Sigma\Sigma^*)^{-1}\widehat{a} - \Xi\}$$

$$+\frac{\theta^2}{2}\{\Sigma^*(\Sigma\Sigma^*)^{-1}\widehat{a} - \Xi\}^*\frac{\partial N_\theta^{-1}}{\partial\theta}\{\Sigma^*(\Sigma\Sigma^*)^{-1}\widehat{a} - \Xi\}$$

Hence, we obtain

$$\begin{aligned}\lim_{\theta\to-\infty}\frac{\chi'(\theta)}{\theta} &= -2(\Lambda N_{-\infty}^{-1}\Xi)^*(\frac{d\bar{q}}{d\theta})_{-\infty} + (\frac{d\bar{q}}{d\theta})_{-\infty}\Lambda N_{-\infty}^{-1}\Lambda^*(\frac{d\bar{q}}{d\theta})_{-\infty} \\ &\quad +\Xi^*N_{-\infty}^{-1}\Xi \\ &= \{\Lambda^*(\frac{d\bar{q}}{d\theta})_{-\infty} - \Xi\}^*N_{-\infty}^{-1}\{\Lambda^*(\frac{d\bar{q}}{d\theta})_{-\infty} - \Xi\}\end{aligned}$$

Asymptotic behavior of $\chi'(\theta)$ as $\theta \to -\infty$

Proposition 5 *If G is stable, then we have*

$$\lim_{\theta \to -\infty} \frac{\chi'(\theta)}{\theta} = \{\Lambda^*(\frac{d\bar{q}}{d\theta})_{-\infty} - \Xi\}^* N_{-\infty}^{-1} \{\Lambda^*(\frac{d\bar{q}}{d\theta})_{-\infty} - \Xi\}$$

where

$$N_{-\infty}^{-1} = I - \Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma$$

Lemma 2 *If G is stable and*

$$\Lambda^*(G^*)^{-1}\{\hat{A}(\Sigma\Sigma^*)^{-1}\Sigma\Xi + g_1\} + \Xi \notin \mathcal{R}(\Sigma^*),$$

then

$$\Lambda^*(\frac{d\bar{q}}{d\theta})_{-\infty} - \Xi \notin \mathcal{R}(\Sigma^*)$$

holds and

$$\{\Lambda^*(\frac{d\bar{q}}{d\theta})_{-\infty} - \Xi\}^* N_{-\infty}^{-1} \{\Lambda^*(\frac{d\bar{q}}{d\theta})_{-\infty} - \Xi\} > 0$$

Remark.

- Previous cases (cf. H-N-S '10, N. '12) $S_T^0 := \int_0^T r(X_s)ds$

$$\inf_{h_\cdot} P(\log \frac{V_T(h)}{S_T^0} \leq -\epsilon T) = 0, \quad \text{however small } \epsilon > 0, \text{ for large } T$$

- Current case

$$\inf_{h_\cdot} P(\log \frac{V_T(h)}{L_T} \leq -KT) > 0, \quad \text{however large } K > 0, \quad \text{for large } T$$

(Implication: much worse performance than the benchmark is possibly expected and the possibility is to be measured)

Explicit representation of $\chi'(0-)$

Besides these condition if B is stable, then

$$\chi'(0-) = \frac{1}{2}\mathrm{tr}[\Lambda\Lambda^*(\frac{d\bar{P}}{d\theta})_{0-}] + b^*(\frac{d\bar{q}}{d\theta})_{0-} + \frac{1}{2}\widehat{a}(\Sigma\Sigma^*)^{-1}\widehat{a} + g_0$$

$$= \frac{1}{2}\mathrm{tr}[\Lambda\Lambda^*(\frac{d\overline{P}}{d\theta})_{0-}] + \frac{1}{2}[\widehat{A}B^{-1}b - \widehat{a}]^*(\Sigma\Sigma^*)^{-1}[\widehat{A}B^{-1}b - \widehat{a}]$$

$$-(B^{-1}b)^* g_1 + g_0$$

$$(\frac{d\overline{P}}{d\theta})_{0-} = \int_0^\infty e^{sB^*}\widehat{A}(\Sigma\Sigma^*)^{-1}\widehat{A}e^{sB}ds$$

5. Under model uncertainty

We consider the situation that random input W_t is not always given as a standard Brownian motion but includes some distortion $Z_t = \int_0^t \zeta_s$, representing model misspecification. Namely, we consider the model spaces $(\Omega, \mathcal{F}, P^{\zeta}; \mathcal{F}_t, W_t)_{\zeta}$, where

$$\left.\frac{dP^{\zeta}}{dP}\right|_{\mathcal{F}_T} = e^{\int_0^T \zeta_s^* dW_s - \frac{1}{2}\int_0^T |\zeta_s|^2 ds}, \qquad \widehat{W}_t = W_t - \int_0^t \zeta_s ds \ : \ \text{B.M. under } P^{\zeta}$$

$$\begin{aligned} \text{State process}: dX_t &= \beta(X_t)dt + \lambda(X_t)dW_t, \quad X_0 = x \in R^n \\ &= \{\beta(X_t) + \lambda(X_t)\zeta_t\}dt + \lambda(X_t)d\widehat{W}_t, \end{aligned}$$

Controlled functional:

$$\begin{aligned} Y_T(h) &= Y_0 + \int_0^T g(X_s, h_s)ds + \int_0^T \{\sigma(X_s)^* h_s - \xi(X_s)\}^* dW_s \\ &= Y_0 + \int_0^T \{g(X_s, h_s) + (\sigma(X_s)^* h_s - \xi(X_s))^* \zeta_s\}ds \\ &\qquad + \int_0^T \{\sigma(X_s)^* h_s - \xi(X_s)\}^* d\widehat{W}_s \end{aligned}$$

Instead of considering under the true model asymptotic behavior of

$$\inf_{h_\cdot} \log P(\frac{1}{T} Y_T(h) \le \kappa), \qquad \text{as } \ T \to \infty,$$

we consider under model uncertainty

$$J(\kappa) := \varliminf_{T\to\infty} \frac{1}{T} \inf_{h \in \mathcal{H}(T)} \sup_{\zeta_\cdot} \log P^{\zeta}(\frac{1}{T}\{Y_T(h) + \frac{\mu}{2}\int_0^T |\zeta_s|^2 ds\} \le \kappa), \tag{5.1}$$

$$\mathcal{H}(T) = \{h(t); R^m\text{- valued, progr. m'ble s.t. } \ P^{\zeta}(\int_0^T |h(s)|^2 ds < \infty) = 1\}$$

• Free parameter μ is supposed to be the certainty level of the model and the problem is concerning asymptotically minimizing the probability under worst case uncertainty at its level. Thus, the supremum is taken.

• Adding the term $\frac{\mu}{2}\int_0^T |\zeta_s|^2 ds$ is naturally understood when we suppose that the right end point of the "effective domain" is ruled by the "robust version of the law of large numbers" as you see later.

In relation to minimizing risk:

(5.2)
$$\inf_{h_\cdot} \sup_{\zeta} \log P^{\zeta}\left(\frac{1}{T}\{Y_T(h) + \frac{\mu}{2}\int_0^T |\zeta_s|^2 ds\} \leq \kappa\right)$$

we are going to study the **lower value** of the game for risk-sensitive control under model uncertainty:

(5.3)
$$\inf_{h_\cdot} \sup_{\zeta} \log E^{\zeta}[e^{\theta\{Y_T(h) + \frac{\mu}{2}\int_0^T |\zeta_s|^2 ds\}}], \qquad \theta < 0.$$

with the dynamics of X_t governed by the stochastic differential equation

(5.4)
$$dX_t = \lambda(X_t)d\widehat{W}_t + \{\beta(X_t) + \lambda(X_t)\zeta_t\}dt, \qquad X_0 = x,$$

Note that

$$\theta\{Y_T(h) + \tfrac{\mu}{2}\int_0^T |\zeta_s|^2 ds\} = \theta y_0 + \theta \int_0^T \{g(X_s, h_s) + (h_s^* \sigma(X_s) - \xi(X_s))\zeta_s\} ds$$

$$+\theta \int_0^T (h_s^* \sigma(X_s) - \xi(X_s) d\widehat{W}_s + \frac{\theta\mu}{2} \int_0^T |\zeta_s|^2 ds$$

For strategy h such that

$$E[e^{\theta \int_0^T h_s^* \sigma(X_s) d\widehat{W}_s - \frac{\theta^2}{2} \int_0^T h_s^* \sigma\sigma(X_s) h_s ds}] = 1$$

measure change

$$\left.\frac{d\widehat{P}^h}{dP^\zeta}\right|_{\mathcal{F}_T} = e^{\theta \int_0^T h_s^* \sigma(X_s) d\widehat{W}_s - \frac{\theta^2}{2} \int_0^T h_s^* \sigma\sigma(X_s) h_s ds}$$

gives us the dynamics of X_t

$$dX_t = \lambda(X_t) d\widehat{W}_t^h + [\beta(X_t) + \lambda(X_t)\zeta_t + \theta\lambda\{\sigma^*(X_t) h_t - \xi(X_t)\}] dt \tag{5.5}$$

with the new Brownian motion: $\widehat{W}_t^h = \widehat{W}_t - \theta \int_0^t \{\sigma(X_s)^* h_s - \xi(X_s)^*\} ds$

The lower H-J-B Isaacs equation of the differential game is

$$\frac{\partial w}{\partial t}+\frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 w]+\frac{1}{2}(\nabla w)^*\lambda\lambda^*\nabla w+\beta^*\nabla w \tag{5.6}$$

$$+\sup_\zeta\inf_h\Lambda_1(x,\nabla w,h,\zeta)=0,$$

where

$$\Lambda(x,p,h,\zeta):=(\lambda\zeta+\theta\lambda\sigma^* h-\theta\lambda\xi)^* p+\theta g(x,h)$$

$$+\theta(h^*\sigma-\xi^*)\zeta+\frac{\theta^2}{2}(\sigma^* h-\xi)^*(\sigma^* h-\xi)+\frac{\theta\mu}{2}|\zeta|^2.$$

We obtain

$$\frac{\partial w}{\partial t}+\frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 w]+\beta^*\nabla w+\theta(1-\tfrac{1}{\theta\mu})\{\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha}-\xi\}^* N^{-1}_{\theta-\frac{1}{\mu}}\lambda^*\nabla w$$

$$+\frac{1}{2}(1-\tfrac{1}{\theta\mu})[(\nabla w)^*\lambda N^{-1}_{\theta-\frac{1}{\mu}}\lambda^*\nabla w+\theta^2\{\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha}-\xi\}^* N^{-1}_{\theta-\frac{1}{\mu}}\{\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha}-\xi\}]$$

$$+\theta\{\tfrac{1}{2}\widehat{\alpha}^*(\sigma\sigma^*)^{-1}\widehat{\alpha}+\widehat{g}_0(x)\}=0$$

Namely,

$$\frac{\partial w}{\partial t} + \tfrac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 w] + \beta^*\nabla w$$

$$+\tfrac{1}{2}\{\lambda^*\nabla w + \theta(\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha} - \xi)\}N_{\theta,\mu}^{-1}\{\lambda^*\nabla w + \theta(\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha} - \xi)\}$$

$$+\theta\{\tfrac{1}{2}\widehat{\alpha}^*(\sigma\sigma^*)^{-1}\widehat{\alpha} + \widehat{g}_0(x)\} = 0$$

$$N_\theta^{-1} = I + \frac{\theta}{1-\theta}\sigma^*(\sigma\sigma^*)^{-1}\sigma$$

$$N_{\theta,\mu}^{-1} := (1 - \frac{1}{\theta\mu})N_{\theta-\frac{1}{\mu}}^{-1} = (1 - \frac{1}{\theta\mu})(I + \frac{\theta - \frac{1}{\mu}}{1-(\theta-\frac{1}{\mu})}\sigma^*(\sigma\sigma^*)^{-1}\sigma)$$

Or,

$$\frac{\partial w}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 w] + \beta^*_{\theta,\mu}\nabla w + \frac{1}{2}(\nabla w)^*\lambda N^{-1}_{\theta,\mu}\lambda^*\nabla w + U_{\theta,\mu} = 0$$

$$N^{-1}_{\theta,\mu} = (1 - \frac{1}{\theta\mu})N^{-1}_{\theta-\frac{1}{\mu}} = (1 - \frac{1}{\theta\mu})(I + \frac{\theta - \frac{1}{\mu}}{1-(\theta-\frac{1}{\mu})}\sigma^*(\sigma\sigma^*)^{-1}\sigma)$$

$$\beta_{\theta,\mu} = \beta + \theta\lambda N^{-1}_{\theta,\mu}\{\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha} - \xi\}$$

$$U_{\theta,\mu} = \theta^2\{\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha} - \xi\}^* N^{-1}_{\theta,\mu}\{\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha} - \xi\}] + \theta\{\frac{1}{2}\widehat{\alpha}^*(\sigma\sigma^*)^{-1}\widehat{\alpha} + \widehat{g}_0(x)\}$$

- $$(1 - \frac{1}{\theta\mu})\frac{1}{1-(\theta-\frac{1}{\mu})}I \;\leq\; N^{-1}_{\theta,\mu} \;\leq\; (1 - \frac{1}{\theta\mu})I$$

• If $N^{-1}_{\theta,\mu}$ is replaced by N^{-1}_{θ} in the above equation, then the HJB equation for the "true model" is recovered. Note that $N^{-1}_{\theta,\mu}$ enjoys almost same property as N^{-1}_{θ}.

Convexity
HJB equation

$$\frac{\partial w}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 w] + \beta^* \nabla w$$

$$+\frac{1}{2}\{\lambda^*\nabla w + \theta(\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha} - \xi)\} N_{\theta,\mu}^{-1}\{\lambda^*\nabla w + \theta(\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha} - \xi)\}$$

$$+\theta\{\frac{1}{2}\widehat{\alpha}^*(\sigma\sigma^*)^{-1}\widehat{\alpha} + \widehat{g}_0(x)\} = 0$$

is rewritten as

$$\frac{\partial w}{\partial t} + \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2 w] + \beta^* D w$$

$$+\sup_{z\in R^M}\{-\frac{1}{2}z^* N_{\theta,\mu} z + z^*(\lambda^* Dw + \theta(\sigma(\sigma\sigma^*)^{-1}\widehat{\alpha} - \xi)\}$$

$$+\frac{\theta}{2}\widehat{\alpha}^*(\sigma\sigma^*)^{-1}\widehat{\alpha} + \theta\widehat{g}_0(x) = 0.$$

This is the H-J-B equation of the stochastic control problem :

$$J = \sup_{Z_\cdot} E[\int_0^T \Phi(X_s, Z_s) ds]$$

with

$$\Phi(x,z) = -\frac{1}{2}z^* N_{\theta,\mu} z + \theta z^*\sigma(\sigma\sigma^*)^{-1}\widehat{\alpha} + \frac{\theta}{2}\widehat{\alpha}^*(\sigma\sigma^*)^{-1}\widehat{\alpha} + \theta\widehat{g}_0(x)$$

and the controlled process governed by

$$dX_t = \lambda(X_t)dW_t + (\beta(X_t) + \lambda(X_t)Z_t)dt$$

Here, note that $\Phi(x, z)$ is convex with respect to θ since

$$N_{\theta,\mu} = \frac{\theta\mu}{\theta\mu - 1}(I - \frac{\theta\mu - 1}{\mu}\sigma^*(\sigma\sigma^*)^{-1}\sigma)$$

and $-N_{\theta,\mu}$ is seen to be convex with respect to θ.

HJB equation of ergodic type

$$\chi_\mu(\theta) = \tfrac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2] + \tfrac{1}{2}(\nabla\bar{w})^*\lambda\lambda^*\nabla\bar{w} + \beta^*\nabla\bar{w}$$

$$+ \sup_\zeta \inf_h \Lambda_1(x, \nabla\bar{w}, h, \zeta) = 0,$$

$$\chi_\mu(\theta) = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\bar{w}] + \beta^*_{\theta,\mu}\nabla\bar{w} + \frac{1}{2}(\nabla\bar{w})^*\lambda N^{-1}_{\theta,\mu}\lambda^*\nabla\bar{w} + U_{\theta,\mu} = 0$$

Remark. The worst case uncertainty is seen to be

$$\tilde{\zeta}(x) = -\frac{1}{\theta\mu}N^{-1}_{\theta-\frac{1}{\mu}}\{\lambda^*\nabla\bar{w}(x) + \theta\sigma^*(\sigma\sigma^*)^{-1}\hat{\alpha}(x) - \theta\xi(x)\}$$

and we see that

$$\| \tilde{\zeta} \|_{L^\infty_{loc}} = O(\frac{1}{\mu})$$

because $\| \nabla\bar{w} \|_{L^\infty_{loc}} \leq C$.

- Solution $(\chi_\mu(\theta), \bar{w})$ to EHJB and its derivative

  - $\chi_\mu(\theta) = \frac{1}{2}\mathrm{tr}[\lambda\lambda^*(x)D^2\bar{w}] + \frac{1}{2}(D\bar{w})^*\lambda\lambda^*(x)D\bar{w}$

    $+\sup_\zeta \inf_{h\in R^m}[\{\beta(x) + \theta\lambda(x)(\sigma^*(x)h - \xi(x))\}^*D\bar{w} + \theta\eta_\mu(x,h,\zeta)],$

$$\eta_\mu(x,h,\zeta) = g(x,h) + \frac{\theta}{2}|\sigma^*h - \xi|^2 + (h^*\sigma - \xi^*)\zeta + \frac{\mu}{2}|\zeta|^2$$

  - $$\chi_\mu(\theta) = \frac{1}{2}\mathrm{tr}[\lambda\lambda^*D^2\bar{w}] + \beta^*_{\theta,\mu}D\bar{w} + \frac{1}{2}(D\bar{w})^*\lambda N^{-1}_{\theta,\mu}\lambda^*D\bar{w} + U_{\theta,\mu}$$

$$\chi_\mu(\theta) = \tilde{L}\bar{w} - \frac{1}{2}(D\bar{w})^*\lambda N^{-1}_{\theta,\mu}\lambda^*D\bar{w} + U_{\theta,\mu}$$

$$\tilde{L}\phi := \frac{1}{2}\mathrm{tr}[\lambda\lambda^*D^2\phi] + \beta^*_{\theta,\mu}D\phi + (D\bar{w})^*\lambda N^{-1}_{\theta,\mu}\lambda^*D\phi \ : \ \text{ergodic}$$

Equation of the formal derivatives of EHJB:

$$(*) \qquad \chi'_\mu(\theta) = \tilde{L}u + \left(\frac{\partial\beta_{\theta,\mu}}{\partial\theta}\right)^*D\bar{v} + \frac{1}{2}(D\bar{v})^*\lambda\frac{\partial N^{-1}_{\theta,\mu}}{\partial\theta}\lambda^*D\bar{v} + \frac{\partial U_{\theta,\mu}}{\partial\theta}$$

$$
(P) \qquad \begin{cases} \rho_\mu(\theta) = \tilde{L}u + (\frac{\partial \beta_{\theta,\mu}}{\partial \theta})^* D\bar{v} + \frac{1}{2}(D\bar{v})^* \lambda \frac{\partial N_{\theta,\mu}^{-1}}{\partial \theta} \lambda^* D\bar{v} + \frac{\partial U_{\theta,\mu}}{\partial \theta} \\ u \in W^{2,p}_{loc}, \quad \sup_{B_R^c} \frac{|u(x)|}{|\bar{w}(x)|} < \infty, \quad \exists R > 0, \end{cases}
$$

under

$$
\sup_{B_R^c} \frac{|f_{\theta,\mu}(x)|}{\tilde{L}\ \bar{w}(x)} < \infty, \quad f_{\theta,\mu}(x) := (\frac{\partial \beta_{\theta,\mu}}{\partial \theta})^* D\bar{w} + \frac{1}{2}(D\bar{w})^* \lambda \frac{\partial N_{\theta,\mu}^{-1}}{\partial \theta} \lambda^* D\bar{w} + \frac{\partial U_{\theta,\mu}}{\partial \theta}
$$

Theorem 3 *Let us assume (2.5)- (2.7) and (2.11)'. Then, we have a solution $(\rho(\theta), u)$ to (P) such that $u \in W^{2,p}_{loc}$, $\sup_{B_R^c} \frac{|u(x)|}{|\bar{w}(x)|} < \infty$, if and only if*

$$
\int \{f_{\theta,\mu}(x) - \rho_\mu(\theta)\} m_{\theta,\mu}(dx) = 0,
$$

where $m_{\theta,\mu}$ is a invariant measure of $\tilde{L}$. Further, this solution u is unique up to additive constants and $\rho_\mu(\theta) = \chi'_\mu(\theta)$.

Theorem 4 *Assume (2.5)-(2.7) and (2.11)'. Then, for* κ *such that* $\chi'_\mu(-\infty) < \kappa < \chi'_\mu(0-)$, *we have*

$$\underline{\lim}_{T\to\infty} \frac{1}{T} \inf_{h_\cdot} \sup_{\zeta_\cdot} \log P^{\zeta}(\frac{1}{T}Y_T(h) + \frac{\mu}{2}\int_0^T |\zeta_s|^2 ds \le \kappa) = -\sup_{\theta<0}\{\theta\kappa - \chi_\mu(\theta)\}.$$

Further, take $\theta(\kappa)$ *which attains the supremum in the right hand side and set* $\overline{h}_t^{(T)} = \widehat{h}(X_t, \nabla w(t, X_t))$ *with*

$$\widehat{h}(x,p) = \frac{(\sigma\sigma^*)^{-1}}{1-\theta}\{\sigma\lambda^*\nabla w(t,x) + \widehat{\alpha}(x) + \sigma(x)(\tilde{\zeta}(x) - \theta\xi(x))\},$$

where $\theta = \theta(\kappa)$ *and* $w(t,x)$ *is the solution to the HJB equation with* $\theta = \theta(\kappa)$. *Then,*

$$\lim_{T\to\infty} \frac{1}{T} \log P^{\tilde{\zeta}}(\frac{1}{T}Y_T(\overline{h}^{(T)}) + \frac{\mu}{2}\int_0^T |\tilde{\zeta}_s|^2 ds \le \kappa) = -\sup_{\theta<0}\{\theta\kappa - \chi_\mu(\theta)\}$$

Characterization of $\chi'_\mu(0-)$

Proposition 6 *Under assumptions (A.1)-(A.5),*

$$\begin{aligned}\chi'_\mu(0-) &= \underline{\lim}_{T\to\infty}\frac{1}{T}\sup_{h_\cdot}\inf_{\zeta_\cdot} E^\zeta[Y_T(h)+\frac{\mu}{2}\int_0^T|\zeta_s|^2ds] \\ &= \underline{\lim}_{T\to\infty}\frac{1}{T}\sup_{h_\cdot}\{-\mu\log E[e^{-\frac{1}{\mu}Y_T(h)}]\}\end{aligned}$$

and $\quad I_\mu(\kappa) := \sup_{\theta<0}\{\theta\kappa-\chi_\mu(\theta)\}=0 \quad$ *for* $\quad \kappa\geq\chi'(0-)$.

Remark. When $Y_T(h)=\log\frac{V_T(h)}{L_T}$, the first line above is **robust log utility maximization** and the second line is risk-sensitive portfolio optimization with risk aversion $-\frac{1}{\mu}$. Their EHJB equation is

$$\widehat{\chi}_\mu = \frac{1}{2}\mathrm{tr}[\lambda\lambda^* D^2\widehat{u}]+\beta^* D\widehat{u}$$

$$-\frac{1}{2\mu}\{\lambda^* D\widehat{u}+(\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha}-\xi)\}^* N_{-\frac{1}{\mu}}^{-1}\{\lambda^* D\widehat{u}+(\sigma^*(\sigma\sigma^*)^{-1}\widehat{\alpha}-\xi)\}$$

$$+\frac{1}{2}\widehat{\alpha}^*(\sigma\sigma^*)^{-1}\widehat{\alpha}+\widehat{g}_0.$$

i.e. $\quad \chi'_\mu(0-)=\widehat{\chi}_\mu.$

Linear Gaussian model

$$\sigma(x) = \Sigma, \;\; \beta(x) = Bx + b, \; \lambda(x) = \Lambda,$$

$$\xi(x) = \Xi, \quad \widehat{\alpha}(x) = \widehat{A}x + \widehat{a}, \quad \widehat{g}_0(x) = g_1^* x + g_0$$

HJB equations

$$(P) \;\; \frac{\partial w}{\partial t} + \frac{1}{2}\mathrm{tr}[\Lambda\Lambda^* D^2 w] + \beta_{\theta,\mu}^* \nabla w + \frac{1}{2}(\nabla w)^* \Lambda N_{\theta,\mu}^{-1} \Lambda^* \nabla w + U_{\theta,\mu} = 0, \quad w(t, x; T) = 0,$$

$$(E) \qquad \chi_\mu(\theta) = \frac{1}{2}\mathrm{tr}[\Lambda\Lambda^* D^2 \bar{w}] + \beta_{\theta,\mu}^* \nabla \bar{w} + \frac{1}{2}(\nabla \bar{w})^* \Lambda N_{\theta,\mu}^{-1} \Lambda^* \nabla \bar{w} + U_{\theta,\mu}$$

Explicit representation by ODEs:

$$w(t, x) = \frac{1}{2}x^* P(t)x + q(t)^* x + l(t), \quad \bar{w}(x) = \frac{1}{2}x^* \bar{P}x + \bar{q}^* x$$

Riccati equation

- $\dot{P}(t) + K_1^* P(t) + P(t) K_1 + P(t) \Lambda N_{\theta,\mu}^{-1} \Lambda^* P(t) + \frac{\theta}{1-(\theta-\frac{1}{\mu})} \widehat{A}^* (\Sigma\Sigma^*)^{-1} \widehat{A} = 0$

$$K_1 = B + \theta \Lambda N_{\theta,\mu}^{-1} \Sigma^* (\Sigma\Sigma^*)^{-1} \widehat{A}, \quad N_{\theta,\mu}^{-1} = (1 - \frac{1}{\theta\mu})(I + \frac{\theta - \frac{1}{\mu}}{1-(\theta-\frac{1}{\mu})} \Sigma^* (\Sigma\Sigma^*)^{-1} \Sigma)$$

- $\dot{q}(t) + (K_1 + \Lambda N_{\theta,\mu}^{-1} \Lambda^* P(t))^* q(t) + P(t) \{ b + \theta \Lambda N_{\theta,\mu}^{-1} (\Sigma^* (\Sigma\Sigma^*)^{-1} \widehat{a} - \Xi) \}$

  $+ \theta^2 \widehat{A}^* (\Sigma\Sigma^*)^{-1} \Sigma N_{\theta,\mu}^{-1} \{ \Sigma^* (\Sigma\Sigma^*)^{-1} \widehat{a} - \Xi \} + \theta (\widehat{A}^* (\Sigma\Sigma^*)^{-1} \widehat{a} + g_1) = 0$

- $\dot{l}(t) + \frac{1}{2} \mathrm{tr}[\Lambda\Lambda^* P(t)] + \{ b + \theta \Lambda N_{\theta,\mu}^{-1} (\Sigma^* (\Sigma\Sigma^*)^{-1} \widehat{a} - \Xi) \}^* q(t)$

  $+ \frac{1}{2} q(t)^* \Lambda N_{\theta,\mu}^{-1} \Lambda^* q(t) + \frac{\theta^2}{2} \{ \Sigma^* (\Sigma\Sigma^*)^{-1} \widehat{a} - \Xi \}^* N_{\theta,\mu}^{-1} \{ \Sigma^* (\Sigma\Sigma^*)^{-1} \widehat{a} - \Xi \}$

  $+ \frac{\theta}{2} \widehat{a}^* (\Sigma\Sigma^*)^{-1} \widehat{a} + \theta g_0 = 0$

Stationary equations:

$$(S1)\qquad \begin{aligned}(K_1+\Lambda N_{\theta,\mu}^{-1}\Lambda^*\bar{P})^*\bar{P}+\bar{P}(K_1+\Lambda N_{\theta,\mu}^{-1}\Lambda^*\bar{P})\\ -\bar{P}\Lambda N_{\theta,\mu}^{-1}\Lambda^*\bar{P}+\frac{\theta}{1-(\theta-\frac{1}{\mu})}\widehat{A}^*(\Sigma\Sigma^*)^{-1}\widehat{A}=0\end{aligned}$$

Under one of conditions $(G),(B),(\Lambda A)$, $P(t;T)\to\bar{P}$, $T\to\infty$, which is the solution to (S1) and $K_1+\Lambda N_{\theta,\mu}^{-1}\Lambda^*\bar{P}$ turns out to be stable in a similar manner to the "true" models.

$$(S2)\qquad \begin{aligned}(K_1+\Lambda N_{\theta,\mu}^{-1}\Lambda^*\bar{P})^*\bar{q}+\bar{P}\{b+\theta\Lambda N_{\theta,\mu}^{-1}(\Sigma^*(\Sigma\Sigma^*)^{-1}\widehat{a}-\Xi)\}\\ +\theta^2\widehat{A}^*(\Sigma\Sigma^*)^{-1}\Sigma N_{\theta,\mu}^{-1}\{\Sigma^*(\Sigma\Sigma^*)^{-1}\widehat{a}-\Xi\}+\theta(\widehat{A}^*(\Sigma\Sigma^*)^{-1}\widehat{a}+g_1)=0\end{aligned}$$

$$(S3)\qquad \begin{aligned}\chi_\mu(\theta)=\frac{1}{2}\mathrm{tr}[\Lambda\Lambda^*\bar{P}]+\{b+\theta\Lambda N_{\theta,\mu}^{-1}(\Sigma^*(\Sigma\Sigma^*)^{-1}\widehat{a}-\Xi)\}^*\bar{q}\\ +\frac{1}{2}\bar{q}^*\Lambda N_{\theta,\mu}^{-1}\Lambda^*\bar{q}+\frac{\theta^2}{2}\{\Sigma^*(\Sigma\Sigma^*)^{-1}\widehat{a}-\Xi\}^*N_{\theta,\mu}^{-1}\{\Sigma^*(\Sigma\Sigma^*)^{-1}\widehat{a}-\Xi\}\\ +\frac{\theta}{2}\widehat{a}^*(\Sigma\Sigma^*)^{-1}\widehat{a}+\theta g_0=0\end{aligned}$$

Asymptotic behavior of $\chi'_\mu(\theta)$ as $\theta \to -\infty$

If $G := B - \Lambda\Sigma^*(\Sigma\Sigma^*)^{-1}\widehat{A}$ is stable,

$$\lim_{\theta\to-\infty} \frac{\chi'_\mu(\theta)}{\theta} = \{\Lambda^*(\frac{d\bar{q}}{d\theta})_{-\infty} - \Xi\}^* N_{-\infty}^{-1} \{\Lambda^*(\frac{d\bar{q}}{d\theta})_{-\infty} - \Xi\}$$

$$N_{-\infty}^{-1} = I - \Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma$$

Moreover, if

$$\Lambda^*(G^*)^{-1}\{\widehat{A}(\Sigma\Sigma^*)^{-1}\Sigma\Xi + g_1\} + \Xi \notin \mathcal{R}(\Sigma^*),$$

then

$$\{\Lambda^*(\frac{d\bar{q}}{d\theta})_{-\infty} - \Xi\}^* N_{-\infty}^{-1} \{\Lambda^*(\frac{d\bar{q}}{d\theta})_{-\infty} - \Xi\} > 0$$

Explicit representation of $\chi'_\mu(0-)$:

$$\chi'_\mu(0-) = \frac{1}{2}\mathrm{tr}[\Lambda\Lambda^*(\frac{d\bar{P}}{d\theta})_{0-}] + b^*(\frac{d\bar{q}}{d\theta})_{0-} + \frac{1}{2}\widehat{a}(\Sigma\Sigma^*)^{-1}\widehat{a} + g_0$$

$$-\{\Lambda^*(\tfrac{d\bar{q}}{d\theta})_{0-} + \Sigma^*(\Sigma\Sigma^*)^{-1}\widehat{a} - \Xi\}^*\widehat{N}_\mu\{\Lambda^*(\tfrac{d\bar{q}}{d\theta})_{0-} + \Sigma^*(\Sigma\Sigma^*)^{-1}\widehat{a} - \Xi\}$$

$$\widehat{N}_\mu = \frac{1}{\mu}\left(I - \frac{\mu}{1+\mu}\Sigma^*(\Sigma\Sigma^*)^{-1}\Sigma\right)$$

This talk is base on

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H. Nagai, in Preparation