Geometric Stopping of a Random Walk and Its Applications to Valuing Equity-linked Death Benefits

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Motivation

- To value guarantees and options in Variable Annuities
- Variable Annuities
  = Investment Funds (Mutual Funds)
  +
  Rider(s) : Guaranteed Minimum Benefits

Mathematical problem

- How to calculate

$$E[v^{(K_x+1)} b(S(K_x), M_S(K_x))].$$

where v is the discount factor per unit time, K_x denotes the curtate future life time r.v., $b(.,.)$ is a benefit function, $S(t)$ denotes the price of a stock or stock index at time t, and $M_S(t)$ is the running maximum of the stock price.

Random walk

- $X(t) = X_1 + \cdots + X_t, \qquad X(0) = 0$
- $X_1, X_2, \ldots$ are i.i.d. r.v.'s, with $\Pr\{X_t = 1\} = p_1$, $\Pr\{X_t = 0\} = p_0$, $\Pr\{X_t = -1\} = p_{-1}$, $p_1 + p_0 + p_{-1} = 1$.

Geometric distribution

Let τ be an independent r.v. with a geometric distribution (say parameter π), such that

$$\Pr\{\tau = t\} = (1-\pi)\pi^t, \qquad t = 0, 1, 2, \ldots . \tag{1}$$

Its probability generating function (pgf) is

$$\mathrm{E}\left[z^\tau\right] = \frac{1-\pi}{1-\pi z}. \tag{2}$$

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The pgf of $X(\tau)$

$$
\begin{aligned}
\mathrm{E}\left[z^{X(\tau)}\right] &= \mathrm{E}\left[\mathrm{E}\left[z^{X(\tau)}|\tau\right]\right] \\
&= \mathrm{E}\left[\mathrm{E}\left[z^{X(1)}\right]^{\tau}\right] \\
&= \mathrm{E}\left[(p_1 z + p_0 + p_{-1} z^{-1})^{\tau}\right] \\
&= \frac{1-\pi}{1-\pi(p_1 z + p_0 + p_{-1} z^{-1})}. \qquad (3)
\end{aligned}
$$

This is a rational function of z, which we can expand by partial fractions. Let

$$0 < \alpha < 1 < \beta < \infty$$

denote the zeros of the denominator in (3), i.e. the solutions of the quadratic equation

$$\pi p_1 z^2 - (1 - \pi p_0) z + \pi p_{-1} = 0. \qquad (4)$$

The pgf of $X(\tau)$

$$\mathrm{E}\left[z^{X(\tau)}\right] = C\frac{\alpha}{z-\alpha} - C\frac{\beta}{z-\beta}, \tag{5}$$

with

$$C = \frac{1-\pi}{\pi}\frac{1}{p(\beta-\alpha)} = \frac{(1-\alpha)(\beta-1)}{\beta-\alpha}. \tag{6}$$

To identify the distribution of $X(\tau)$, we rewrite (5) as

$$\mathrm{E}\left[z^{X(\tau)}\right] = C\frac{\alpha/z}{1-\alpha/z} + C\frac{1}{1-z/\beta}. \tag{7}$$

The distribution of $X(\tau)$

$$
\begin{aligned}
\Pr\{X(\tau) = j\} &= C\alpha^{-j}, \qquad j = -1, -2, ..., \\
\Pr\{X(\tau) = j\} &= C\beta^{-j}, \qquad j = 0, 1, 2, ... \,. \qquad (8)
\end{aligned}
$$

Thus $X(\tau)$ has a two-sided geometric distribution.

The record highs and lows of the random walk

- $$M(t) = \max\{0, X(1), ..., X(t)\} \tag{9}$$

  denote the running maximum and, similarly, $m(t)$ the running minimum after t steps.

Joint distribution in the trinomial tree model

Suppose $X(t) = X_1 + ... + X_t$,

where X_i takes three values: $-1,\ 0,\ 1$

and $P(X_i = 1) = P(X_i = -1) = p/2,\ P(X_i = 0) = q$ with $p + q = 1$.

We assume that $X_1, X_2, ...$ is an i.i.d. sequence. Since the random walk $X(t)$ is symmetric, the reflection principle is true (the proof is the same as that for simple symmetric random walk).

$$P(\{X(t) = j, M(t) \geq k\}) = P(X(t) = 2k - j)$$

Joint distribution in the trinomial tree model

Now we assume that the random walk is not symmetric,

that is $P(X_i = 1) = p_1,\ P(X_i = -1) = p_{-1},\ P(X_i = 0) = p_0$, with $p_1 + p_{-1} + p_0 = 1$. Let

$$\Lambda_t = \left(\frac{1}{2(p_{-1}p_1)^{1/2} + p_0}\right)^t (p_{-1}/p_1)^{X(t)/2}.$$

Then Λ_t is a martingale under P and $E[\Lambda_t] = 1$.

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Joint distribution in the trinomial tree model

Let $dQ/dP|_{\mathcal{F}_t} = \Lambda_t$, where $\mathcal{F}_t = \sigma\{X_1, ..., X_t\}$.

Then under Q, $X_1, ..., X_t, ...$ is i.i.d symmetric random walk;

$$Q(X_1 = 1) = E_Q[I_{(X_1=1)}] = E_P[\Lambda_1 I_{(X_1=1)}] = \frac{(p_{-1}p_1)^{1/2}}{2(p_{-1}p_1)^{1/2} + p_0}$$

,

similarity,

$$Q(X_1 = -1) = \frac{(p_{-1}p_1)^{1/2}}{2(p_{-1}p_1)^{1/2} + p_0}$$

,

Joint distribution in the trinomial tree model

$$Q(X_1 = 0) = \frac{p_0}{2(p_{-1}p_1)^{1/2} + p_0}$$

.

$X_1, X_2, ...$ are independent under Q, because

$$\Lambda_t = \prod_{i=1}^{t}[\frac{(p_{-1}p_1)^{X_i/2}}{2(p_{-1}p_1)^{1/2} + p_0}].$$

Joint distribution in the trinomial tree model

$$P(X(t) = j, M(t) \geq k) = E_P[I_A],$$

where $A = \{X(t) = j, M(t) \geq k\}$.

$$E_P[I_A] = E_Q[I_A \Lambda_t^{-1}] = (2(p_{-1}p_1)^{1/2} + p_0)^t (p_1/p_{-1})^{j/2} Q(A), \quad (10)$$

and $Q(A) = Q(X(t) = 2k - j)$ because, under Q, $X_1, X_2, ...$ is a symmetric random walk.

Joint distribution in the trinomial tree model

$$
\begin{aligned}
& Q(X(t)=2k-j) = E_Q[I_{(X(t)=2k-j)}] = E_P[\Lambda_t I_{(X(t)=2k-j)}] \\
= \; & (2(p_{-1}p_1)^{1/2}+p_0)^{-t}(p_{-1}/p_1)^{(2k-j)/2}P(X(t)=2k-j). \qquad (11)
\end{aligned}
$$

From (10) and (11)

$$
P(X(t)=j, M(t)\geq k) = (p_{-1}/p_1)^{k-j}P(X(t)=2k-j).
$$

Joint distribution in the trinomial tree model

Since τ is independent of $X(t)$, we have

$$P(X(\tau)=j, M(\tau) \geq k) = (p_{-1}/p_1)^{k-j} P(X(\tau)=2k-j).$$

From this we can obtain the joint probability function of $X(\tau)$ and $M(\tau)$.

Distributions of $M(\tau)$ and $m(\tau)$

Both $M(\tau)$ and $m(\tau)$ have geometric distributions:

$$\Pr\{M(\tau) \geq k\} = \beta^{-k}, \qquad k = 0, 1, 2, ..., \tag{12}$$
$$\Pr\{m(\tau) \leq k\} = \alpha^{-k}, \qquad k = 0, -1, -2, ..., \tag{13}$$

or,

$$\Pr\{M(\tau) = k\} = (\beta - 1)\beta^{-k-1}, \qquad k = 0, 1, 2, ..., \tag{14}$$
$$\Pr\{m(\tau) = k\} = (1 - \alpha)\alpha^{-k}, \qquad k = 0, -1, -2, ... \tag{15}$$

We note that $M(\tau) \geq k$ is the event that $X(n)$ reaches level k before or at time τ. Similarly, $m(\tau) \leq k$ is the event that $X(n)$ falls to level k before or at time τ.

Proof

Let k_1 be a negative integer and k_2 a positive integer. Let Π_1 be the probability that $X(t)$ reaches k_1 before or at time τ, and before $X(t) = k_2$. Similarly, Let Π_2 be the probability that $X(t)$ reaches k_2 before or at time τ, and before $X(t) = k_1$. It is verified that

$$\{z^{X(t)} I_{(\tau \geq t)};\ t = 0, 1, 2, ...\} \tag{16}$$

is a martingale for $z = \alpha$ and $z = \beta$.

Proof

Now we stop each of the two martingales the first time (before or at time τ) when $X(t) = k_1$, or $X(t) = k_2$, or else at time $\tau + 1$. Then the optional sampling theorem yields the equations

$$\Pi_1 \alpha^{k_1} + \Pi_2 \alpha^{k_2} = 1, \tag{17}$$

$$\Pi_1 \beta^{k_1} + \Pi_2 \beta^{k_2} = 1. \tag{18}$$

It follows that

$$\Pi_1 = \frac{\beta^{k_2} - \alpha^{k_2}}{\alpha^{k_1}\beta^{k_2} - \alpha^{k_2}\beta^{k_1}}, \tag{19}$$

$$\Pi_2 = \frac{\alpha^{k_1} - \beta^{k_1}}{\alpha^{k_1}\beta^{k_2} - \alpha^{k_2}\beta^{k_1}}. \tag{20}$$

In the limit $k_1 \to -\infty$ ($k_2 = k$), (20) becomes (12). Similarly, in the limit $k_2 \to \infty$ ($k_1 = k$), (19) becomes (13).

The distributions of $M(\tau)$ and $X(\tau) - m(\tau)$ are the same

We note that for each t, the r.v.'s $M(t)$ and $X(t) - m(t)$ have the same distribution. This follows from

$$\begin{aligned} M(t) &= \max\{0, X_1, X_1 + X_2, \ \dots, X_1 + X_2 + \dots + X_t\}, \\ X(t) - m(t) &= \max\{0, X_t, X_t + X_{t-1}, \ \dots, X_t + X_{t-1} + \dots + X_1\}. \end{aligned}$$

Hence, the distributions of $M(\tau)$ and $X(\tau) - m(\tau)$ are the same; their probabilities are given by (14). Similarly, the distributions of $m(\tau)$ and $X(\tau) - M(\tau)$ are the same; their probabilities are given by (15).

The r.v.'s $M(\tau)$ and $X(\tau) - M(\tau)$ are independent

Because the conditional distribution of $X(\tau) - M(\tau)$, given $M(\tau)$, is the conditional distribution of $X(\tau)$, given $X(n) \leq 0$ for $n = 1, ..., \tau$, and hence the same for all values of $M(\tau)$. To see this, consider the first time t when $X(t) = M(\tau)$; thus $\tau \geq t$ and $X(n) - X(t) \leq 0$ for $n = t, ..., \tau$. Then observe that the conditional distribution of $\tau - t$ does not depend on t.

The r.v.'s $M(\tau)$ and $X(\tau) - M(\tau)$ are independent

$$
\begin{aligned}
& P(X(\tau) - M(\tau) = k | M(\tau)) \\
= \; & P(X(\tau) - M(\tau) = k | M(\tau) = X(t^*), \tau > t^*) \\
= \; & P(X(\tau) - X(t^*) = k | X(t) \leq X(t^*), t = 0, 1, ..., \tau, \tau > t^*) \\
= \; & P(X(\tau) - X(t^*) = k | X(t) \leq X(t^*), t = t^*, t^* + 1, ..., \tau, \tau > t^*) \\
= \; & P(X(\tau - t^*) = k | X(t) - X(t^*) \leq 0, t = t^*, ..., \tau, \tau > t^*) \\
= \; & P(X(\tau - t^*) = k | X(t - t^*) \leq 0, t = t^*, ..., \tau, \tau > t^*) \\
= \; & P(X(\tau^*) = k | X(0) \leq 0, ... X(\tau^*) \leq 0),
\end{aligned}
$$

where τ^* and τ have the same distribution.

The joint distribution of $X(\tau)$ and $M(\tau)$

$$
\begin{aligned}
&\Pr\{X(\tau) = j, M(\tau) = h\} \\
&= \Pr\{M(\tau) - X(\tau) = h - j,\ M(\tau) = h\} \\
&= \Pr\{M(\tau) - X(\tau) = h - j\}\Pr\{M(\tau) = h\} \\
&= \Pr\{m(\tau) = -(h - j)\}\Pr\{M(\tau) = h\} \\
&= (1 - \alpha)\alpha^{h-j}(\beta - 1)\beta^{-h-1}. \qquad (21)
\end{aligned}
$$

Similarly, for $h = 0, -1, -2, ...$ and $j \geq h$, we have

$$
\begin{aligned}
&\Pr\{X(\tau) = j, m(\tau) = h\} \\
&= \Pr\{X(\tau) - m(\tau) = j - h,\ m(\tau) = h\} \\
&= \Pr\{X(\tau) - m(\tau) = j - h\}\Pr\{m(\tau) = h\} \\
&= \Pr\{M(\tau) = j - h\}\Pr\{m(\tau) = h\} \\
&= (\beta - 1)\beta^{-(j-h)-1}(1 - \alpha)\alpha^{-h}. \qquad (22)
\end{aligned}
$$

For $k = 0, 1, 2, ...$ and $j \le k$, we find that

$$\Pr\{X(\tau) = j, M(\tau) \ge k\} = C\alpha^{-j}(\frac{\alpha}{\beta})^k. \tag{23}$$

Of course for $j \ge k$,

$$\Pr\{X(\tau) = j, M(\tau) \ge k\} = \Pr\{X(\tau) = j\} = C\beta^{-j}. \tag{24}$$

Similarly, for $k = 0, -1, -2, ...$ and $j \ge k$ one shows that

$$\Pr\{X(\tau) = j, m(\tau) \le k\} = C\beta^{-j}(\frac{\beta}{\alpha})^k. \tag{25}$$

Of course

$$\Pr\{X(\tau) = j, m(\tau) \le k\} = \Pr\{X(\tau) = j\} = C\alpha^{-j} \tag{26}$$

for $j \le k$.

Remark

The proofs that $M(\tau)$ and $X(\tau) - M(\tau)$ are independent, and that $X(\tau) - m(\tau)$ has the same distribution as $m(\tau)$, are valid for a general random walk. It follows that

$$\begin{aligned} P_{X(\tau)}(z) &= \mathrm{E}[z^{X(\tau)}] = \mathrm{E}[z^{M(\tau)+X(\tau)-M(\tau)}] \\ &= \mathrm{E}[z^{M(\tau)}] \times \mathrm{E}[z^{X(\tau)-M(\tau)}] \\ &= \mathrm{E}[z^{M(\tau)}] \times \mathrm{E}[z^{m(\tau)}] = P_{M(\tau)}(z) P_{m(\tau)}(z). \quad (27) \end{aligned}$$

This formula is a version of the *Wiener-Hopf factorization*.

Remark

If X_1 takes integer values from $-n$ to $+m$, then

$$\begin{aligned} P_{X(\tau)}(z) &= \frac{1-\pi}{1-\pi P_{X_1}(z)]} = \frac{1-\pi}{1-\pi\sum_{j=-n}^{m} p_j z^j} \\ &= \frac{(1-\pi)z^n}{g(z)}, \end{aligned} \tag{28}$$

where $g(z) = \left(1-\pi\sum_{j=-n}^{m} p_j z^j\right) z^n$ is a polynomial of degree $m+n$.

Remark

Because $1 > \pi = \pi \sum_{j=-n}^{m} p_j$, we have

$$|z^n| > |g(z) - z^n|, \qquad \text{for} \quad |z| = 1. \tag{29}$$

Then by Rouché's Theorem, $g(z)$ has the same number of zeros inside the complex disk of radius 1 as the function z^n. Denote these n zeros of $g(z)$ as $\alpha_1, ..., \alpha_n$. Denote the other zeros of $g(z)$, those with absolute value greater than 1, as $\beta_1, ..., \beta_m$. Then, the pgf $P_{X(\tau)}(z)$ is proportional to

$$\begin{aligned} & \frac{z^n}{\left(\prod_{j=1}^{n}(z - \alpha_j)\right)\left(\prod_{j=1}^{m}(z - \beta_j)\right)} \\ = \quad & \frac{1}{\left(\prod_{j=1}^{n}(1 - \alpha_j/z)\right)\left(\prod_{j=1}^{m}(z - \beta_j)\right)}. \end{aligned} \tag{30}$$

Remark

Note that $P_{M(\tau)}(1) = 1$ and $P_{m(\tau)}(1) = 1$. Because $M(\tau) \geq 0$, $P_{M(\tau)}(z)$ exists for each z with $|z| < 1$. Similarly, $P_{m(\tau)}(z)$ exists for each z with $|z| > 1$. Therefore

$$P_{M(\tau)}(z) = \frac{\prod_{j=1}^{m}(\beta_j - 1)}{\prod_{j=1}^{m}(\beta_j - z)}, \quad P_{m(\tau)}(z) = \frac{\prod_{j=1}^{n}(1 - \alpha_j)}{\prod_{j=1}^{m}(1 - \alpha_j/z)}. \quad (31)$$

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Remark

In the special case of a simple random walk, from (5) and (6), we have

$$P_{X(\tau)}(z) = C\frac{(\beta-\alpha)z}{(z-\alpha)(\beta-z)} = \frac{\beta-1}{\beta-z} \times \frac{(1-\alpha)z}{z-\alpha}. \tag{32}$$

Indeed, the first factor of the RHS of (32) is the pgf of (14), and the second factor is the pgf of (15).

Applications

Let K_x denote the curtate future life time r.v. (measured in number of time units). Upon death, a benefit payment B is payable at time $K_x + 1$. We assume a constant rate of interest and denote by v the discount factor per unit time. The general goal is to calculate

$$E[v^{K_x+1}B] = \sum_{n=0}^{\infty} \Pr\{K_x = n\} v^{n+1} E[B|K_x = n], \tag{33}$$

the expectation of the discounted benefit payment.
We assume that K_x and $\{S(t);\ t = 0, 1, ...\}$ are independent.

Reduced problem

The distribution of K_x can be approximated by a combination of geometric distributions:

$$\Pr\{K_x = n\} \approx \sum_{j=1}^{m} c_j (1 - \pi_j) \pi_j^n, \tag{34}$$

where $c_1, ..., c_m, \pi_1, ..., \pi_m$ are suitably chosen. As a consequence, it suffices to consider the reduced problem, where K_x in (33) is a geometrically distributed r.v. τ, that is, to calculate

$$\mathrm{E}[v^{\tau+1} B] = \sum_{n=0}^{\infty} (1 - \pi) \pi^n v^{n+1} \mathrm{E}[B | \tau = n]. \tag{35}$$

Reduced problem

We can simplify the problem by one more step:

$$\begin{aligned} \mathrm{E}[v^{\tau+1}B] &= \frac{v(1-\pi)}{1-v\pi}\sum_{n=0}^{\infty}(1-v\pi)(v\pi)^n \mathrm{E}[B|\tau=n] \\ &= \frac{v(1-\pi)}{1-v\pi}\tilde{\mathrm{E}}[B] = \mathrm{E}[v^{\tau+1}]\tilde{\mathrm{E}}[B]. \end{aligned} \tag{36}$$

Formally, the discounting has been eliminated in the second expectation. But note that it is taken with respect to the geometric distribution with parameter $v\pi$ instead of π.

The trinomial tree model

We assume the trinomial tree model, that is, that for some $a > 1$,

$$S(t) = S(0)a^{X(t)}, \qquad t = 0, 1, 2, ..., \tag{37}$$

where $\{X(t);\ t = 0, 1, 2, ...\}$ is the random walk taking three possible values.

$0 < \alpha < 1 < \beta < \infty$ are now defined as the solution of the equation

$$\pi p_1 z^2 - (1 - \pi p_0) z + \pi p_{-1} = 0. \tag{38}$$

$B = S(\tau)$

By (36) and (3) with $z = a$, we have

$$\begin{aligned} \mathrm{E}[v^{\tau+1} S(\tau)] &= \frac{v(1-\pi)}{1-v\pi} \tilde{\mathrm{E}}[S(\tau)] \\ &= \frac{v(1-\pi)}{1-v\pi} S(0) \tilde{\mathrm{E}}[a^{X(\tau)}] \\ &= S(0) \frac{v(1-\pi)}{1-v\pi(p_1 a + p_0 + p_{-1} a^{-1})}. \end{aligned} \tag{39}$$

B a function of $S(K_x)$

$$B = b(S(K_x)) \tag{40}$$

for some function $b(s)$. Because of the factorization formula (36), we are to calculate

$$\begin{aligned} \mathrm{E}[B] &= \mathrm{E}[b(S(\tau))] \\ &= \sum_{j=-\infty}^{\infty} b(S(0)a^j)\Pr\{X(\tau) = j\} \\ &= \sum_{j=-\infty}^{-1} b(S(0)a^j)C\alpha^{-j} + \sum_{j=0}^{\infty} b(S(0)a^j)C\beta^{-j} \end{aligned} \tag{41}$$

by (8). Note that $\mathrm{E}[B]$ is a function of $S(0)$; we shall use the symbol $\mathcal{E}_b(S(0))$ for it.

Out-of-the money put option

For a *put option* with exercise price K, $b(s) = (K - s)_+$. Then

$$\mathrm{E}[(K - S(\tau))_+] = \sum_{j=-\infty}^{\ell} (K - S(0)a^j)\Pr\{X(\tau) = j\}, \qquad (42)$$

where ℓ is the greatest integer j such that $K - S(0)a^j > 0$. For the *out-of-the money case*, $S(0) > K$ $(\ell < 0)$, we find that

$$\mathrm{E}[(K - S(\tau))_+] = C\left[K\frac{(1/\alpha)^\ell}{1-\alpha} - S(0)\frac{(a/\alpha)^\ell}{1-\alpha/a}\right], \qquad (43)$$

where C is given by (6).

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Out-of-the money call option

For a *call option* with exercise price K, $b(s) = (s - K)_+$. In the *out-of-the money case*, $S(0) \le K$ ($\ell \ge 0$), we obtain

$$\begin{aligned} \mathrm{E}[(S(\tau) - K)_+] &= \sum_{j=\ell+1}^{\infty} (S(0)a^j - K) C \beta^{-j} \\ &= C \left[S(0) \frac{(a/\beta)^{\ell+1}}{1 - a/\beta} - K \frac{(1/\beta)^{\ell+1}}{1 - 1/\beta} \right]. \quad (44) \end{aligned}$$

Put-call-parity

From

$$(K - S(\tau))_+ - (S(\tau) - K)_+ = K - S(\tau), \tag{45}$$

we see that

$$\mathrm{E}[(K - S(\tau))_+] - \mathrm{E}[(S(\tau) - K)_+] = K - \mathrm{E}[(S(\tau))], \tag{46}$$

with

$$\mathrm{E}[S(\tau)] = S(0)\frac{1-\pi}{1-\pi(pa+qa^{-1})}. \tag{47}$$

In-the money put option

$$
\begin{aligned}
& \mathrm{E}[(K - S(\tau))_+] = \mathrm{E}[(S(\tau) - K)_+] + K - \mathrm{E}[(S(\tau))] \\
= \quad & C\left[S(0)\frac{(a/\beta)^{\ell+1}}{1-a/\beta} - K\frac{(1/\beta)^{\ell+1}}{1-1/\beta}\right] + K - \mathrm{E}[(S(\tau))], \quad (48)
\end{aligned}
$$

In-the money call option

$$
\begin{aligned}
& \mathsf{E}[(S(\tau)-K)_+] = \mathsf{E}[(K-S(\tau))_+] - K + \mathsf{E}[(S(\tau))] \\
= \quad & C\left[K\frac{(1/\alpha)^\ell}{1-\alpha} - S(0)\frac{(a/\alpha)^\ell}{1-\alpha/a}\right] - K + \mathsf{E}[(S(\tau))], \qquad (49)
\end{aligned}
$$

Expiry at time T

Suppose that $b(S(K_x))$ is payable at time $K_x + 1$, provided that $K_x < T$, where T is a positive integer. Because of (36), we want to determine

$$E[b(S(\tau))I_{(\tau<T)}] = E[b(S(\tau))] - E[b(S(\tau))I_{(\tau\geq T)}], \quad (50)$$

where I denotes the indicator function. Using the memoryless property of τ and the independence between τ and the stock price process, we see that

$$\begin{aligned} E[b(S(\tau))I_{(\tau\geq T)}] &= \Pr\{\tau \geq T\}E[b(S(\tau))|\tau \geq T] \\ &= \pi^T E[\mathcal{E}_b(S(T))], \end{aligned} \quad (51)$$

where $\mathcal{E}_b(s) = E[b(S(\tau))|S(0) = s]$ has been defined

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The remaining task is to calculate

$$\begin{aligned}
&\mathrm{E}[\mathcal{E}_b(S(T))] \\
&= \mathrm{E}[\mathcal{E}_b(S(T)) I_{(S(T) \le K)}] + \mathrm{E}[\mathcal{E}_b(S(T)) I_{(S(T) > K)}]. \quad (52)
\end{aligned}$$

For $b(s) = I_{(s \leq K)}$,

$$\begin{aligned} \mathsf{E}[\mathcal{E}_b(S(T))] &= C\frac{(1/\alpha)^{\ell}}{1-\alpha}\mathsf{E}[\alpha^{X(T)}]\Pr\{S(T) > K; \alpha\} \\ &\quad + \Pr\{S(T) \leq K\} \\ &\quad - C\frac{(1/\beta)^{\ell+1}}{1-1/\beta}\mathsf{E}[\beta^{X(T)}]\Pr\{S(T) \leq K; \beta\}. \end{aligned}$$

Esscher transform

We have used the formula

$$E[h^X I_A] = E[h^X]\Pr\{A; h\}, \qquad h > 0, \tag{53}$$

where the probability of the event A is calculated with respect to the Esscher transformed probabilities,

$$\Pr\{X = j; h\} = h^j \Pr\{X = j\}/E[h^X]. \tag{54}$$

Formula (53) has been used for $X = X(T)$ and $h = \beta, \alpha, a$.

Furthermore, we note that the generating function of the transformed probabilities is

$$\mathrm{E}[z^X; h] = \mathrm{E}[(hz)^X]/\mathrm{E}[h^X], \tag{55}$$

which can be useful to determine the transformed probabilities.

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Barrier options

First we consider a double barrier option with barriers $L < U$ and initial stock price $S(0)$ with $L < S(0) < U$. We may assume that

$$k_1 = \log_a(\frac{L}{S(0)}), \quad k_2 = \log_a(\frac{U}{S(0)}) \qquad (56)$$

are integers.

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Double barrier options

The probability that the stock price first hits the lower end L of the barrier interval, and that this takes place before or at time τ, is Π_1 given by (19). At that time, because of the memoryless property of the geometric distribution, the conditional expectation of B is $\mathcal{E}_b(L)$; here $\mathcal{E}_b(s)$ refers to the ordinary option. Similarly, the probability that the stock price first hits the upper end U of the barrier interval, and that this happens before or at time τ, is Π_2 given by (20). Then the conditional expectation of B is $\mathcal{E}_b(U)$.

Double barrier options

Hence, by conditioning we find that

$$E[B] = \Pi_1 \mathcal{E}_b(L) + \Pi_2 \mathcal{E}_b(U). \tag{57}$$

Single barrier options

The *down-and-in* option becomes alive if the stock price hits the level L before or at time τ, the probability of which is given by (13) with $k = k_1$. Thus

$$\mathrm{E}[B] = \alpha^{-k_1}\mathcal{E}_b(L). \tag{58}$$

In contrast, the *up-and-in* option becomes alive if the stock price hits the level U before or at time τ, the probability of which is given by (12) with $k = k_2$. Thus

$$\mathrm{E}[B] = \beta^{-k_2}\mathcal{E}_b(U). \tag{59}$$

Knock-out barrier options

A *knock-out* option is alive at time 0 and is knocked out when the stock price hits the predetermined barrier(s). The payoff B of a knock-out option is the payoff of the corresponding ordinary option reduced by the payoff of the corresponding knock-in option. From this and (57) - (59) it follows that

$$\mathrm{E}[B] = \mathcal{E}_b(S(0)) - \Pi_1 \mathcal{E}_b(L) - \Pi_2 \mathcal{E}_b(U) \tag{60}$$

for the knock-out double barrier option,

$$\mathrm{E}[B] = \mathcal{E}_b(S(0)) - \Pi_1 \mathcal{E}_b(L) \tag{61}$$

for the *down-and-out* option, and

$$\mathrm{E}[B] = \mathcal{E}_b(S(0)) - \Pi_2 \mathcal{E}_b(U) \tag{62}$$

for the *up-and-out* option.

Remark

For the single barrier options the expected payoff can also be obtained from (23) - (26). For example, for the up-and-in option, it is

$$\sum_{j=-\infty}^{\infty} b(S(0)a^j)\Pr\{X(\tau)=j, M(\tau)\geq k_2\}$$

$$= \sum_{j=-\infty}^{k_2} b(S(0)a^j)C\alpha^{-j}(\frac{\alpha}{\beta})^{k_2} + \sum_{j=k_2+1}^{\infty} b(S(0)a^j)C\beta^{-j}. \quad (63)$$

To see that this is the same as (59), use (41) and note that

$$\mathcal{E}_b(U) = \mathcal{E}_b(S(0)a^{k_2}).$$

Out-of-the-money fixed strike lookback call option

The payoff is

$$[S(0)a^{M(\tau)} - K]_+, \tag{64}$$

whose time-0 value, by (14), is

$$\begin{aligned} E[[S(0)a^{M(\tau)} - K]_+] &= \sum_{k=\ell+1} [S(0)a^k - K](\beta - 1)\beta^{-k-1} \\ &= S(0)\frac{\beta - 1}{\beta}\frac{(a/\beta)^{\ell+1}}{1 - a/\beta} - K(1/\beta)^{\ell+1}. \end{aligned} \tag{65}$$

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In-the-money fixed strike lookback call option

The payoff is

$$\max(H, S(0)a^{M(\tau)}) - K. \tag{66}$$

By rewriting (66) as

$$H - K + [S(0)a^{M(\tau)} - H]_+ \tag{67}$$

and using (65) with K replaced by H, we find that the time-0 value of (66) is

$$H - K + S(0)\frac{\beta - 1}{\beta}\frac{(a/\beta)^{\ell+1}}{1 - a/\beta} - H(1/\beta)^{\ell+1}. \tag{68}$$

Floating strike lookback put option

The payoff at time τ is

$$\max(H, \max_{0 \le t \le \tau} S(t)) - S(\tau), \tag{69}$$

where $H \ge S(0)$. By comparing (69) with (66), we see that its time-0 value is (68) but with K replaced by $\mathrm{E}[S(\tau)]$. The result is

$$H - \mathrm{E}[S(\tau)] + S(0)\frac{\beta - 1}{\beta}\frac{(a/\beta)^{\ell+1}}{1 - a/\beta} - H(1/\beta)^{\ell+1}. \tag{70}$$