Valuation of American partial barrier options

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Partial Barrier Option

- Underlying price is monitored during only part of the option's lifetime.
- Forward starting barrier options where the barrier appears at a fixed date strictly after the option's initial starting date.
- Early ending barrier options where the barrier disappears at a specified date strictly before the expiry date.
- Heynan and Kat(1994) gave valuation formulas for partial barrier options in terms of bivariate normal distribution functions.

American Option

- American options give their holders the additional flexibility of early exercise
- An exact and closed-form pricing solution of American option has not existed because the option price and the early exercise boundary must be determined simultaneously.
- Literature of American options has proposed only numerical solution methods and analytical approximations.
- Numerical methods : finite difference method, binomial model

American Barrier Option (Ingersoll(1988))

- Ingersoll (1998) presented American up-and-in put price by an approximation method based on barrier options using constant and exponential exercise policies.
- Advantage in simplicity and speed

American Barrier Option (Ingersoll(1988))

- $D(S, t\ ; A)$: value at time t of receiving one dollar at time T if the event A occurs.(digital or binary option)
- $DS(S, t\ ; A)$: value at time t of receiving one share of stock at time T if the event A occurs.(digital share)
- $E(S, t, K_\tau\ ; A)$: value at time t of payment $X - K_\tau$ at the first time τ that the stock price S hits the barrier K_τ provided the event A occurs before time T, where X is a strike price.(first-touch digital)

American Up-and-In Put Option (Ingersoll(1988))

- T : expiration , X : strike price , U : up-barrier , K^* : optimal exercise policy
- τ_{B_1} : first time the stock price is equal to B_1
- $\tau_{B_1 B_2}$: first time after τ_{B_1} that the stock price is equal to B_2
- $A_1 = \{t < \tau_U < T,\ \tau_{UK^*} > T,\ S_T < X\}$: event of exercise at maturity under the optimal policy
- $A_2 = \{t < \tau_U, \tau_{UK^*} < T\}$: event of early exercise under the optimal policy

Value of the Up-and-In Put

$$UIP = X \cdot D(S, t\ ; A_1) - DS(S, t\ ; A_1) + E(S, t, K^*\ ; A_2)$$

American Up-and-In Put Option (Ingersoll(1988))

For constant exercise policies k,

$$UIP \geq UIP_{\text{const}} = \max_{k} \left[X \cdot D(S,t\ ; A_3) - DS(S,t\ ; A_3) + E(S,t,k\ ; A_4) \right]$$

where $A_3 = \{t < \tau_U < T,\ \tau_{Uk} > T,\ S_T < X\}$,
$A_4 = \{t < \tau_U, \tau_{Uk} < T\}$.

$$D(S,t\ ; A_3) = e^{-r(T-t)} \left\{ \left(\frac{U}{S_t}\right)^{\frac{2\mu}{\sigma^2}} \left[N\left(h_1\left(\frac{U^2}{S_t k}\right)\right) - N\left(h_1\left(\frac{U^2}{S_t X}\right)\right) \right] + \left(\frac{k}{U}\right)^{\frac{2\mu}{\sigma^2}} \left[N\left(h_1\left(\frac{S_t k^2}{U^2 X}\right)\right) - N\left(h_1\left(\frac{S_t k}{U^2}\right)\right) \right] \right\}$$

American Up-and-In Put Option (Ingersoll(1988))

$$DS(S,t\ ;A_3)$$
$$= S_t e^{-q(T-t)}\left\{\left(\frac{U}{S_t}\right)^{\frac{2\overline{\mu}}{\sigma^2}}\left[N\left(h_2\left(\frac{U^2}{S_t k}\right)\right)-N\left(h_2\left(\frac{U^2}{S_t X}\right)\right)\right]\right.$$
$$\left.+\left(\frac{k}{U}\right)^{\frac{2\overline{\mu}}{\sigma^2}}\left[N\left(h_2\left(\frac{S_t k^2}{U^2 X}\right)\right)-N\left(h_2\left(\frac{S_t k}{U^2}\right)\right)\right]\right\}$$

$$E(S,t,k\ ;A_4)$$
$$=(X-k)\left[\left(\frac{k}{S_t}\right)^{b-\beta}\left(\frac{k}{U}\right)^{2\beta}N\left(g_1\left(\frac{S_t k}{U^2}\right)\right)\right.$$
$$\left.+\left(\frac{k}{S_t}\right)^{b+\beta}\left(\frac{U}{k}\right)^{2\beta}N\left(-g_1\left(\frac{U^2}{S_t k}\right)\right)\right]$$

American Up-and-In Put Option (Ingersoll(1988))

where N is the standard normal distribution function,

$$h_1(z) = \frac{\ln z + \mu(T-t)}{\sigma\sqrt{T-t}}, \quad h_2(z) = \frac{\ln z + \overline{\mu}(T-t)}{\sigma\sqrt{T-t}},$$

$$g_1(z) = \frac{\ln z + \beta\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$\mu = r - q - \frac{1}{2}\sigma^2, \quad \overline{\mu} = r - q + \frac{1}{2}\sigma^2, \quad b = \frac{\mu}{\sigma^2},$$

$$\text{and} \quad \beta = \sqrt{b^2 + \frac{2r}{\sigma^2}}\,.$$

American Partial Barrier Option

Consider the up-and-in put where the barrier appears at a specified time T_1.

- If the underlying asset price hits the up-barrier over the time period $[T_1, T]$, then the put option can be exercised before or at time T with strike price X.
- If the asset price never crosses the up-barrier in $[T_1, T]$, this option pays off zero.
- Barrier option of American type where the barrier appears at a fixed date strictly after the option's initial starting date.
- Numerical methods such as Monte Carlo method and Trinomial lattice model for American partial barrier option demand much time.

American Partial Barrier Option

Value of partial up-and-in put is

$$PUIP = X \cdot D(S,t\,;A_5) - DS(S,t\,;A_5) + E(S,t,K^*\,;A_6)$$

where

- $\tau_{U(T_1)}$: first time that the stock price reaches the barrier U after time T_1
- $\tau_{Uk(T_1)}$:first time that the stock price falls to the exercise policy k after $\tau_{U(T_1)}$
- $A_5 = \{\tau_{U(T_1)} < T,\ \tau_{UK^*(T_1)} > T,\ S_T < X\}$: event of exercise at maturity under the optimal policy
- $A_6 = \{\tau_{UK^*(T_1)} < T\}$: event of early exercise under the optimal policy

American Partial Barrier Option

Value of partial up-and-in put is

$$PUIP_{\text{const}} = \max_{k \in \mathcal{K}_c} \left[X \cdot D(S,t\,;A_7) - DS(S,t\,;A_7) + E(S,t,k\,;A_8) \right]$$

where $\mathcal{K}_c$ is the set of all constant functions.

- $A_7 = \{\tau_{U(T_1)} < T,\ \tau_{Uk(T_1)} > T,\ S_T < X\}$: event of exercise at maturity under a constant policy k
- $A_8 = \{\tau_{Uk(T_1)} < T\}$: event of early exercise under policy k

$$D(S,t;A_7)$$

$$= e^{-r(T-t)} \left(\frac{U}{S_t}\right)^{\frac{2\mu}{\sigma^2}} [G_1(X) - G_1(k)] + e^{-r(T-t)} \left(\frac{k}{S_t}\right)^{\frac{2\mu}{\sigma^2}} [G_2(X) - G_2(k)]$$

$$+ e^{-r(T-t)} \left(\frac{k}{U}\right)^{\frac{2\mu}{\sigma^2}} [G_3(X) - G_3(k)] + e^{-r(T-t)} [G_4(X) - G_4(k)],$$

$$DS(S,t;A_7)$$

$$= S_t e^{-q(T-t)} \left(\frac{U}{S_t}\right)^{\frac{2\overline{\mu}}{\sigma^2}} \left[\overline{G}_1(X) - \overline{G}_1(k)\right] + S_t e^{-q(T-t)} \left(\frac{k}{S_t}\right)^{\frac{2\overline{\mu}}{\sigma^2}} \left[\overline{G}_2(X) - \overline{G}_2(k$$

$$+ S_t e^{-q(T-t)} \left(\frac{k}{U}\right)^{\frac{2\overline{\mu}}{\sigma^2}} \left[\overline{G}_3(X) - \overline{G}_3(k)\right] + S_t e^{-q(T-t)}[\overline{G}_4(X) - \overline{G}_4(k)]$$

where

$$G_1(X) = N_2\left(h_3\left(\frac{U}{S_t}\right), -h_1\left(\frac{U^2}{S_t X}\right); -\sqrt{\frac{T_1-t}{T-t}}\right), G_2(X) = N_2\left(-h_3\left(\frac{U}{S_t}\right), h_1\left(\frac{k^2}{S_t X}\right); -\sqrt{\frac{T_1-t}{T-t}}\right)$$

$$G_3(X) = N_2\left(-h_3\left(\frac{S_t}{U}\right), h_1\left(\frac{S_t k^2}{U^2 X}\right); -\sqrt{\frac{T_1-t}{T-t}}\right), G_4(X) = N_2\left(h_3\left(\frac{S_t}{U}\right), -h_1\left(\frac{S_t}{X}\right); -\sqrt{\frac{T_1-t}{T-t}}\right)$$

$$h_1(z) = \frac{\ln z + \mu(T-t)}{\sigma\sqrt{T-t}}, \quad h_3(z) = \frac{\ln z + \mu(T_1-t)}{\sigma\sqrt{T_1-t}}, \quad \overline{G}_i(X) : \mu \ in \ G_i(X) \ \rightarrow \mu = r - q + \frac{\sigma^2}{2}$$

$$E(S,t,k;A_8) = (X-k)\left[\left(\frac{U}{k}\right)^{\beta-b}\left(\frac{U}{S_t}\right)^{\beta+b} H_1(k) + \left(\frac{k}{S_t}\right)^{\beta+b} H_2(k) + \left(\frac{S_t}{U}\right)^{\beta-b}\left(\frac{k}{U}\right)^{\beta+b} H_3(k) + \left(\frac{S_t}{k}\right)^{\beta-b} H_4(k)\right]$$

where

$$H_1(k) = N_2\left(g_2\left(\frac{U}{S_t}\right), -g_1\left(\frac{U^2}{S_t k}\right); -\sqrt{\frac{T_1-t}{T-t}}\right),$$

$$H_2(k) = N_2\left(-g_2\left(\frac{U}{S_t}\right), g_1\left(\frac{k}{S_t}\right); -\sqrt{\frac{T_1-t}{T-t}}\right),$$

$$H_3(k) = N_2\left(-g_2\left(\frac{S_t}{U}\right), g_1\left(\frac{S_t k}{U^2}\right); -\sqrt{\frac{T_1-t}{T-t}}\right),$$

$$H_4(k) = N_2\left(g_2\left(\frac{S_t}{U}\right), -g_1\left(\frac{S_t}{k}\right); -\sqrt{\frac{T_1-t}{T-t}}\right), \quad g_2(z) = \frac{\ln z + \beta\sigma^2(T_1-t)}{\sigma\sqrt{T_1-t}}$$

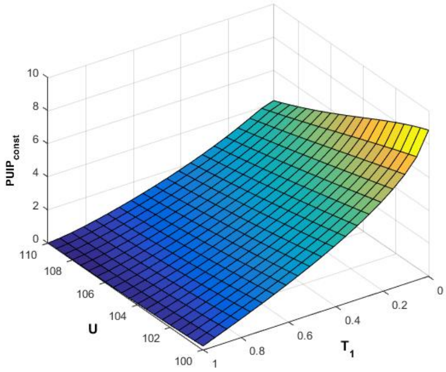

Figure 1: $PUIP_{\text{const}}$ Result, varying U and T_1 when $U \geq X$ ($S_0 = 100,\ X = 100,\ r = 0.05,\ \sigma = 0.3$, and $T = 1$)

Table 1: Comparison of American Partial Barrier Put Option Values $PUIP_{\mathrm{const}}$ with varying S_0 and strike price X

S_0	X	V(10)	V(50)	V(100)	V(500)	MC	AMM5	k^*
	95	1.5969	1.6001	1.6007	1.6007	1.6042	1.6018	78.9070
	97.5	2.0753	2.0792	2.0796	2.0797	2.0918	2.0815	80.6617
100	100	2.6477	2.6523	2.6523	2.6525	2.6504	2.6553	82.3900
	102.5	3.3213	3.3261	3.3261	3.3261	3.3026	3.3304	84.1115
	105	4.1018	4.1062	4.1062	4.1064	4.0656	4.1125	85.8060

- $U = 105,\ T_1 = 0.1,\ T = 0.5,\ \sigma = 0.3,\ r = 0.05,\ q = 0.$
- $V(N)$: option value of $PUIP_{\mathrm{const}}$, N : number of constant policy barriers evenly spaced from 0 to strike price X
- MC : Monte Carlo simulation (Antithetic Variates, a Variance Reduction Method)
- AMM5 : Trinomial lattice method (Adaptive Mesh Model with level 5)
- k^* : optimal policy barrier for V(10000)

Table 2: Comparison of American Partial Barrier put option values $PUIP_{c}onst$ with varying U and T_1

U	T_1	V(10)	V(50)	V(100)	V(500)	MC	AMM5	k^*
102	0.1	5.3817	5.3837	5.3851	5.3852	5.3874	5.3743	85.3230
	0.3	2.5479	2.5603	2.5610	2.5611	2.5616	2.5567	89.5545
104	0.1	4.5022	4.5061	4.5061	4.5065	4.5101	4.4840	85.6485
	0.3	1.9848	1.9950	1.9950	1.9952	1.9782	1.9721	89.8800
106	0.1	3.7235	3.7282	3.7282	3.7282	3.7152	3.7320	85.9635
	0.3	1.5514	1.5593	1.5593	1.5594	1.5496	1.5569	90.1950

$$(S_0 = 100,\ X = 105,\ \sigma = 0.3,\ T = 0.5,\ r = 0.05,\ q = 0)$$

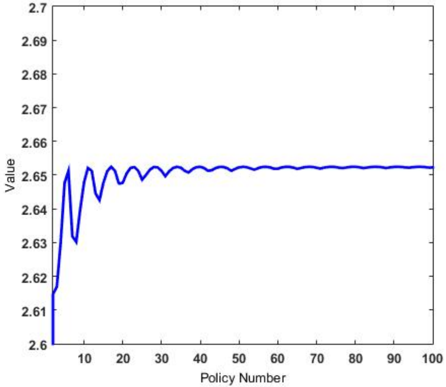

Figure 2: $PUIP_{\mathrm{const}}$ Result, varying policy barrier number N ($S_0 = 100,\ X = 100,\ U = 105,\ r = 0.05,\ \sigma = 0.3,\ T_1 = 0.1$, and $T = 0.5$)