Risk and Ambiguity in Asset Returns
– Cross-Sectional Differences –

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Outline

Ambiguity in asset markets

- Explore implications of ambiguity and ambiguity aversion on portfolio choices and asset returns.
- Motivated to explain some phenomena that cannot be explained by expected utility functions.
- We concentrate on the composition of stocks in optimal portfolios. Cf. Chen and Epstein (2002) and Epstein and Miao (2003).

Our "stocks": FF6 portfolios

1. Sort out the stocks traded on NYSE, AMEX, and NASDAQ in terms of the market equity (market value, market capitalization) and the ratio of the book equity (book value) to the market equity.
2. Partition them into six groups, according to whether the ME belongs to the top or bottom 50%, and whether the BE/ME belongs to the top or bottom 30%, or neither.
3. Form the ME-weighted portfolio for each of the six groups:

	Bottom 50% of ME	Top 50% of ME
Bottom 30% of BE/ME	SL	BL
Middle 40% of BE/ME	SN	BN
Top 30% of BE/ME	SH	BH

Return on the FF6 portfolios

The means, variances, and covariances of the monthly returns in % of the FF6 portfolios, and the mean of the risk-free rates, from 1926 to 2014.

	Mean (%)	SL	SN	SH	BL	BN	BH
risk-free	0.28						
SL	0.98	57.36	50.77	55.76	34.61	35.62	43.74
SN	1.28	50.77	49.64	55.73	31.89	35.74	45.07
SH	1.48	55.76	55.73	67.64	34.64	41.20	53.89
BL	0.91	34.61	31.89	34.64	28.62	27.43	31.63
BN	0.97	35.62	35.74	41.20	27.43	32.89	38.32
BH	1.19	43.74	45.07	53.89	31.63	38.32	50.95

The Small and High portfolios have higher means and variances.

Mean-variance-efficient portfolio and market portfolio

The proportions of the total investment allocated to the FF6 portfolios.

	MVE portfolio	MKT portfolio
SL	−3.3641	0.0246
SN	3.4532	0.0295
SH	1.1213	0.0208
BL	1.8397	0.5074
BN	−1.0806	0.3120
BH	−0.9694	0.1057
Total	1.0000	1.0000

The MVE portfolio involves large long and short positions.

Introducing ambiguity to rationalize the market portfolio

- In the CARA-normal setting, the investor would hold a MVE portfolio.
- For what kind of utility functions is the MKT portfolio optimal?
- We use the ambiguity-averse utility functions of Klibanoff, Marinnacci, and Mukerji (2005).
- In particular, we extend the CARA-normal setting to the case where the expected asset returns are ambiguous but the covariance matrix is not, and the second-order belief of expected asset returns is also a multivariate normal distribution.

Our results: Old and new

- We proved that for every portfolio, there is an ambiguity-averse investor for whom the portfolio is optimal if and only if the expected rate of return of the portfolio exceeds the risk-free rate.
- For each such portfolio, we identified a class of minimally ambiguity-averse investors for whom it is optimal.
- We propose a couple of notions of, and find, the least ambiguity-averse investor among them.
- We investigate whether the representative investor is reasonably ambiguity-averse based on the FF6 portfolios.

Outline

Ambiguity and ambiguity aversion

- Represent the returns of N assets by a random vector X.
- Denote the risk-free rate by R.
- Conditional on a random vector M, X has mean vector M: $X|M \sim \mathcal{N}(M, \Sigma_{X|M})$.
- Suppose that $M \sim \mathcal{N}(\mu_M, \Sigma_M)$. It is the second-order belief.
- An ambiguity-averse utility function $U_{\gamma,\theta}$ is defined by

$$U_{\gamma,\theta}\left(a^\top X + bR\right) = E\left[u_\gamma\left(u_\theta^{-1}\left(E\left[u_\theta\left(a^\top X + bR\right)|M\right]\right)\right)\right],$$

where u_γ and u_θ have CARA γ and θ. If $\gamma > \theta$, then $U_{\gamma,\theta}$ is ambiguity-averse.

Optimal portfolio

- The utility function $U_{\gamma,\theta}$ can be rewritten as

$$u_\gamma^{-1}(U_{\gamma,\theta}(a^\top X + bR)) = \mu_M^\top a + Rb - \frac{\theta}{2} a^\top \Sigma_{X|M} a - \frac{\gamma}{2} a^\top \Sigma_M a.$$

Cf. Maccheroni, Marinacci, and Ruffino (2013)

- The first-order condition for an optimal portfolio is

$$\mu_M - R\mathbf{1} = (\theta \Sigma_{X|M} + \gamma \Sigma_M) a = \theta(\Sigma_X + \eta \Sigma_M) a$$

$$\text{thus, } a = \frac{1}{\theta}(\Sigma_X + \eta \Sigma_M)^{-1}(\mu_M - R\mathbf{1}), \quad (1)$$

where $\eta = \gamma/\theta - 1$ and $\Sigma_X = \Sigma_{X|M} + \Sigma_M$.

Role of ambiguity in asset composition

- The optimal portfolio a is a scalar multiple of the MVE portfolio $(\mathbf{1}^\top \Sigma_X^{-1}(\mu_M - R\mathbf{1}))^{-1}\Sigma_X^{-1}(\mu_M - R\mathbf{1})$ when $\eta\Sigma_M = 0$.
- It is so even when $\Sigma_M = \lambda\Sigma_X$ for some $\lambda \in [0, 1]$. Indeed, then,

$$a = \frac{1}{\theta(1+\lambda\eta)}\Sigma_X^{-1}(\mu_M - R\mathbf{1}).$$

- It is so as long as $\Sigma_M a = \lambda\Sigma_X a$ for some $\lambda \in [0, 1]$.
- The expected excess return is always strictly positive:

$$a^\top(\mu_M - R\mathbf{1}) = \frac{1}{\theta}(\mu_M - R\mathbf{1})^\top(\Sigma_X + \eta\Sigma_M)^{-1}(\mu_M - R\mathbf{1}) > 0.$$

The converse also holds

We take Σ_X as objective and observable, and Σ_M as subjective and unobservable; and so is the decomposition $\Sigma_X = \Sigma_{X|M} + \Sigma_M$.

Theorem 1. For every portfolio $a \in \boldsymbol{R}^N$, if $a^\top(\mu_M - R\mathbf{1}) > 0$, then there is a (Σ_M, η, θ) for which (1) holds.

- We have characterized the set of all such (Σ_M, η, θ)'s by finding:
  1. the supremum $\bar{\theta}$ of the coefficients of risk aversion equal to $a^\top(\mu_M - R\mathbf{1})/(a^\top \Sigma_X a)$; and
  2. for each $\theta \in (0, \bar{\theta})$, a unique $(\Sigma_M^\theta, \eta^\theta)$ that are smaller than any other (Σ_M, η) such that (Σ_M, η, θ) belongs to the set.
- With the data of FF6 portfolios,

$$\min_{\theta \in (0, \bar{\theta})} \eta^\theta = 9.31.$$

Cf. $\varphi(y) = (u_\gamma \circ u_\theta^{-1})(y) = -(-y)^{\eta+1}$.

Outline

Another criterion of reasonable parameter values

- Let a be the MKT portfolio and choose a rationalizing (Σ_M, η, θ).
- Then, we decompose the expected excess returns into two parts

$$\mu_M - R\mathbf{1} = (\text{Risk Part}) + (\text{Ambiguity Part})$$

  Cf. Chen and Epstein (2002), Ui (2011), and Thimme and Völkert (2015).
- We (wish to) find a "minimal" ambiguity part by varying (Σ_M, η, θ).
- It depends only on the data of asset markets.
- Valid even when ambiguity and ambiguity aversion cannot be separated.

"Equilibrium" interpretation of the decomposition

- The Risk Part is the expected excess return that would induce the investor to hold the MKT portfolio if the ambiguity were completely removed and the covariance matrix of asset returns were $\Sigma_X - \Sigma_M$.
- The Ambiguity Part is the expected excess return that would induce the investor to hold the MKT portfolio if the pure risk were completely removed and the covariance matrix of asset returns were Σ_M.

This decomposition depends on (Σ_M, η, θ). Among all the (Σ_M, η, θ)'s with which the market portfolio a is optimal, we wish to know the one that "minimizes" the second term.

Our approach

- We minimize the ambiguity part (only) over all $(\Sigma_M^\theta, \eta^\theta, \theta)$'s, defined after Theorem 1.
- For $(\Sigma_M^\theta, \eta^\theta)$, the risk-ambiguity decomposition of asset returns is

$$\mu_M - R\mathbf{1} = \Sigma_X v_1^\theta + \Sigma_X v_2^\theta,$$

where v_1^θ is a purely risky portfolio and v_2^θ is the purely ambiguous portfolio, in the sense that $\Sigma_M v_1^\theta = 0$ and $\Sigma_M v_2^\theta = \Sigma_X v_1^\theta$.
- Our minimization problem is

$$\inf_{\theta \in (0,\bar{\theta})} \left(\left(\Sigma_X v_2^\theta \right)^\top \Sigma_X^{-1} \left(\Sigma_X v_2^\theta \right) \right)^{1/2} = \inf_{\theta \in (0,\bar{\theta})} \left((v_2^\theta)^\top \Sigma_X v_2^\theta \right)^{1/2}.$$

Solution of our minimization problem

Theorem 2. $\left((v_2^\theta)^\top \Sigma_X v_2^\theta\right)^{1/2}$ is a strictly decreasing function of θ. Moreover, $\Sigma_X v_1^{\bar{\theta}} = \bar{\theta}\Sigma_X a$.

- The minimization problem is "solved" at $\theta = \bar{\theta}$. Moreover, since $a^\top \Sigma_X v_1^{\bar{\theta}} = a^\top(\mu_M - R\mathbf{1})$, the expected excess return of the market portfolio a can be explained completely by the risk part.
- It can be shown that

$$\left((v_2^\theta)^\top \Sigma_X v_2^\theta\right)^{1/2} = \underbrace{\text{Sharpe ratio}}_{\frac{\text{mean}}{\text{standard deviation}}} \quad \text{of} \quad \left(\frac{1}{\theta}\Sigma_X^{-1}(\mu_M - R\mathbf{1}) - a\right)$$

Numerical result based on FF6 portfolios

When the ambiguity part is minimized, the risk-ambiguity decomposition of returns are as follows:

	μ_M	$\mu_M - R\mathbf{1}$	risk part	ambiguity part
SL	0.98	0.69	0.84	−0.15
SN	1.28	0.98	0.81	0.17
SH	1.48	1.19	0.92	0.27
BL	0.91	0.63	0.65	−0.02
BN	0.97	0.69	0.70	−0.01
BH	1.19	0.92	0.83	0.09
MKT	0.98	0.70	0.70	0.00

The returns of the Small and High portfolios are more ambiguous.
Cf. Bossaerts, Ghirardato, Guarnaschelli, and Zame (2010).

Another numerical result

When the coefficient of ambiguity aversion is minimized ($\eta^\theta = 9.31$), the risk-ambiguity decomposition of returns are as follows:

	μ_M	$\mu_M - R\mathbf{1}$	risk part	ambiguity part
SL	0.98	0.69	0.44	0.25
SN	1.28	0.98	0.39	0.60
SH	1.48	1.19	0.43	0.77
BL	0.91	0.63	0.33	0.30
BN	0.97	0.69	0.35	0.34
BH	1.19	0.92	0.41	0.51
MKT	0.98	0.70	0.35	0.35

The High portfolios are more ambiguous, but the Small ones are not.

Outline

Conclusion

- Extended the CARA-Normal setup to accommodate ambiguity.
- Discussed some criteria with respect to which the investor is "reasonably" ambiguity-averse.
- Assessed to what extent the representative investor is ambiguity-averse based on the U.S. equity market data.

- Should spell out pros and cons of various criteria in view of "portability" and applications.
- Should separate the issue of ambiguity distribution across different asset classes from that of reasonable ambiguity aversion.