The Sustainable Black-Scholes Equations

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Outline

Introduction

- In incomplete markets, a basic Black-Scholes perspective has to be complemented by the valuation of market imperfections.
- "Black-Scholes Ponzi schemes": always more derivatives need to be issued for remunerating the capital attracted by the already opened positions.
- Sustainable Black-Scholes equations for a portfolio of options:

  trade additive Black-Scholes price + nonlinear funding cost + cost of remunerating at a hurdle rate the residual risk left by imperfect hedging.

Cost of Capital

- In presence of hedging imperfections resulting in a nonvanishing loss (and profit) process ϱ of the bank, a conditional risk measure $\mathrm{EC} = \mathrm{EC}_t(\varrho)$ must be dynamically computed and reserved by the bank as Economic Capital.

- Albanese, Caenazzo and Crépey '16 (Section 5): the capital valuation adjustment (KVA) needed by the bank in order to remunerate its shareholders for their capital at risk at some average hurdle rate h (e.g. 10%) at any point in time in the future is:

$$\mathrm{KVA} = \mathrm{KVA}_t(\varrho) = h\mathbb{E}_t\left[\int_t^T e^{-(r+h)(s-t)}\mathrm{EC}_s(\varrho)ds\right]. \tag{1}$$

Cost of Funding

- r_t: a risk-free OIS short term interest rate and $\beta_t = e^{-\int_0^t r_s ds}$ be the corresponding risk-neutral discount factor.

- The bank can invest at the risk-free rate r but can only obtain unsecured funding at a shifted rate $r + \lambda > r$.

  This entails funding costs over OIS and a related funding valuation adjustment (FVA) for the bank.

- Focus on capital and funding: we ignore counterparty risk for simplicity, so that λ is interpreted as a pure funding liquidity basis.

Notations and assumptions

- The bank "marks to the model" its derivative portfolio, assumed bought from the client at time 0, by means of an FVA-deducted value process Θ.

- The bank may also set up a (possibly imperfect) hedge $(-\eta)$ in the hedging assets, for some predictable row-vector process η of the same dimension as the vector gain process $\mathcal{M}$ of unit positions held in the hedging assets.

- The depreciation of Θ, the funding expenditures and the loss $\eta d\mathcal{M}$ on the hedge, minus the option payoffs as they mature, are instantaneously realized into the loss(-and-profit) process ϱ of the bank. At any time t, the amount on the funding account of the bank is Θ_t.

- The economic capital can be used by the trader for her funding purposes provided she pays to the shareholders the OIS rate on EC that they would make otherwise by depositing it (assuming it all cash for simplicity).

Notations and assumptions

- The value process Θ of the trade already includes the FVA as a deduction, but ignores the KVA, which is considered as a risk adjustment computed in a second step.
- In other words, the trader's account and the KVA account are kept separate from each other.

Loss(-and-profit) process ϱ of the bank

$$d\varrho_t = \underbrace{-\sum_i \omega_i (S_{T_i} - K_i)^+ \delta_{T_i}(dt)}_{\text{call payoffs}}$$

$$+ \underbrace{r_t \mathrm{EC}_t(\varrho) dt}_{\text{payment of internal lending of the EC funding source at OIS rate}}$$

$$+ \underbrace{\Big((r_t + \lambda_t)(\Theta_t - \mathrm{EC}_t(\varrho))^+ - r_t(\Theta_t - \mathrm{EC}_t(\varrho))^- \Big) dt}_{\text{portfolio funding costs/benefits}}$$

$$+ \underbrace{(-d\Theta_t)}_{\text{depreciation of } \Theta} + \underbrace{\eta_t d\mathcal{M}_t}_{\text{loss on the hedge}}$$

$$= -d\Theta_t - \sum_i \omega_i (S_{T_i} - K_i)^+ \delta_{T_i}(dt) + \Big(\lambda_t (\Theta_t - \mathrm{EC}_t(\varrho))^+ + r_t \Theta_t \Big) dt + \eta_t d\mathcal{M}_t.$$

FVA-deducted value process Θ

A no-arbitrage condition that the loss process ϱ of the bank should follow a risk-neutral martingale (assuming integrability) and the terminal condition $\Theta_T = 0$ lead to the following FVA-deducted risk-neutral valuation BSDE:

$$\Theta_t = \underbrace{\mathbb{E}_t\Big[\sum_{t<T_i} \beta_t^{-1}\beta_{T_i}\omega_i(S_{T_i} - K_i)^+\Big]}_{\Theta_t^{(0)}} - \underbrace{\mathbb{E}_t\Big[\int_t^T \beta_t^{-1}\beta_s\lambda_s\big(\Theta_s - \mathrm{EC}_s(\varrho)\big)^+ ds\Big]}_{\mathrm{FVA}_t}. \quad (3)$$

We treat option pay-offs as cash-flows at their maturity times rather than a terminal condition in the equations, in particular $\Theta_T = 0$.

The funding source provided by economic capital creates a feedback loop from EC into FVA, which makes the FVA smaller.

Special cases and Nonstandard BSDEs

- Case 1: $\lambda = 0$, then, whatever the hedge η, Θ reduces to $\Theta^{(0)}$, which corresponds to the usual trade additive (linear) no-arbitrage pricing formula for a portfolio of options, with zero FVA, but with a KVA given by (1), depending on the hedge η.

- Case 2: $\lambda \neq 0$, if there exists a replicating hedge η, i.e. $\eta = \eta^\star$ such that the ensuing ϱ is constant, i.e. $\eta^\star_t d\mathcal{M}_t$ coincides with the martingale part of $\Theta^\star$, then the resulting ϱ, EC and KVA vanish (since we assumed $\mathrm{EC}(0) = 0$) and the ensuing FVA-deducted value process is given by $\Theta^\star$, the solution the following backward SDE:

$$\Theta^\star_t = \mathbb{E}_t\Big[\sum_{t<T_i} \beta_t^{-1}\beta_{T_i}\omega_i(S_{T_i} - K_i)^+ - \int_t^T \beta_t^{-1}\beta_s\lambda_s(\Theta^\star_s)^+ ds\Big] \tag{4}$$

- Apart from the above special cases where $\lambda = 0$ or $\eta = \eta^\star$, the BSDE (3) for Θ is nonstandard due to the term $\mathrm{EC} = \mathrm{EC}_t(\varrho)$ in the FVA. At time t, $\mathrm{EC}_t(\varrho)$ may depend on the whole future of the process (ρ_s), $s \geq t$.

Markovian Black-Scholes Setup

- We assume a constant risk-free rate r and a Black-Scholes stock S with volatility σ and constant dividend yield q.
- The risk-neutral martingale $\mathcal{M}$: $d\mathcal{M}_t = dS_t - (r-q)S_t dt$.
  $\mathcal{A}_S^{bs} = (r-q)S\partial_S + \frac{1}{2}\sigma^2 S^2 \partial^2_{S^2}$ the risk-neutral Black-Scholes generator.
- In the Black-Scholes setup and assuming a stylized Markovian specification

$$\mathrm{EC}_t(\varrho) = f\sqrt{\frac{d\langle\varrho\rangle}{dt}} \tag{5}$$

  (the stylized VaR which is proportional to the instantaneous volatility of the loss process ϱ with a "quantile level" f) as well as $\lambda = \lambda(t, S_t)$, $\eta_t = \eta(t, S_t)$.

Given a tentative FVA-deducted price process of the form $\Theta_t = u(t, S_t)$ for some to-be-determined function $u = u(t, S)$, we have

$$\sqrt{\frac{d\langle\varrho\rangle}{dt}} = \sigma S_t \big|\partial_S u(t, S_t) - \eta(t, S_t)\big|. \tag{6}$$

PDE

Accordingly, let the function u be defined by $u_i(t, S)$ on each strip $(T_{i-1}, T_i] \times (0, \infty)$, where $(u_i)_{1 \le i \le n}$ is the unique sequence of classical solutions, to the following PDE cascade, for i decreasing from n to 1 (closing the system by setting $u_{n+1} = 0$ and $T_0 = 0$):

$$\begin{cases} u_i(T_i, S) = u_{i+1}(T_i, S) + \omega_i(S - K_i)^+ \text{ on } (0, \infty) \\ \partial_t u_i + \mathcal{A}_S^{bs} u_i - \lambda\big(u_i - f\sigma S|\partial_S u_i - \eta|\big)^+ - ru_i = 0 \text{ on } [T_{i-1}, T_i) \times (0, \infty). \end{cases} \tag{7}$$

Markovian Black-Scholes Setup

We set $\eta = (1-\alpha)\partial_S u$, where α in $[0, 100\%]$ is the mis-hedge parameter, then $\sqrt{\frac{d\langle\varrho\rangle}{dt}} = \alpha\sigma S_t \big|\partial_S u(t, S_t)\big|$ and the FVA$= \Theta^{(0)} - \Theta$ and KVA processes are

$$\begin{aligned}
\mathrm{FVA}_t(\varrho) &= \mathbb{E}_t\Big[\int_t^T e^{-r(s-t)}\lambda_s\big(u(s,S_s) - \alpha f\sigma S_s\big|\partial_S u(s,S_s)\big|\big)^+ ds\Big] \\
&= v(t,S_t) = u_{bs}(t,S_t) - u(t,S_t), \qquad (8)\\
\mathrm{KVA}_t(\varrho) &= h\mathbb{E}_t\Big[\int_t^T e^{-(r+h)(s-t)}\alpha f\sigma S_s\big|\partial_S u(s,S_s)\big| ds\Big] = w(t,S_t),
\end{aligned}$$

where u_{bs} is the trade additive Black-Scholes portfolio value and where the FVA and KVA pricing functions v and w satisfy

$$\begin{cases}
v(T,S) = w(T,S) = 0 \text{ on } (0,\infty) \\
\partial_t v + \mathcal{A}_S^{bs} v + \lambda\big(u_{bs} - v - \alpha f\sigma S|\Delta_{bs} - \partial_S v|\big)^+ - rv = 0 \text{ on } [0,T)\times(0,\infty) \\
\partial_t w + \mathcal{A}_S^{bs} w + \alpha h f\sigma S|\Delta_{bs} - \partial_S v| - (r+h)w = 0 \text{ on } [0,T)\times(0,\infty),
\end{cases} \qquad (9)$$

in which $\Delta_{bs} = \partial_S u_{bs}$.

Sustainable Black-Scholes Equations

These "sustainable Black-Scholes PDEs" (9) allow computing an FVA and KVA deducted price

$$u - w = u_{bs} - v - w$$

that would be sustainable for the bank even in the limit case of a portfolio held on a run-off basis, with no new trades ever entered in the future.

Numerical example

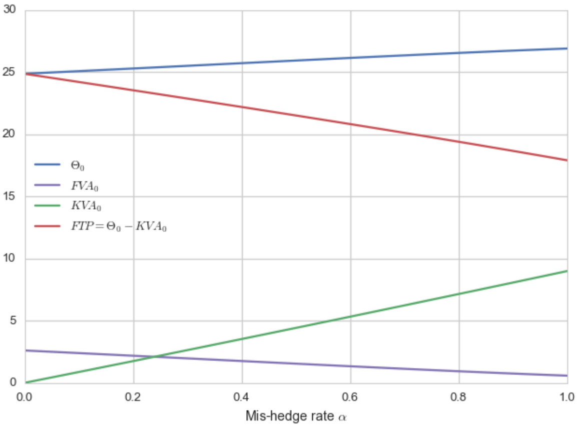

Figure : XVAs and FTP as a function of the mis-hedge parameter α: Without model uncertainty.

UVM

- AVA (additional valuation adjustment): the magnifying impact of model risk on the different XVA metrics.

- We assess model risk from the angle of Avellaneda, Levy and Paras 1995's uncertain volatility model (UVM).

- $d\mathcal{M}_t := \sigma_t S_t dW_t = dS_t - (r-q)S_t dt$, where $\sigma_t \in [\underline{\sigma}, \overline{\sigma}]$ for every t.

Second order BSDEs

Due to the model uncertainty, the trader values it $\Theta_0 = \inf_{\mathbb{Q}\in\mathcal{Q}} \Theta_0^{\mathbb{Q}}$.

The corresponding equation for the FVA-deducted value Θ would appear as

$$\Theta_t = \operatorname*{ess\,inf}_{\mathbb{Q}'\in\mathcal{Q}(t,\mathbb{Q},\mathbb{F}_+)} \mathbb{E}_t^{\mathbb{Q}'}\Big[\sum_{t<T_i} \beta_t^{-1}\beta_{T_i}\omega_i(S_{T_i}-K_i)^+ - \int_t^T \beta_t^{-1}\beta_s\lambda_s\big(\Theta_s - \mathrm{EC}_s^{\mathbb{Q}'}(\varrho)\big)^+ ds\Big], \quad t\in[0,T], \quad \mathbb{Q}-a.s.. \tag{10}$$

In the Markovian setting with VaR-like specification of Economic Capital, we can make rigorous statements.

According to the second order BSDEs theory, the PDE (7) becomes:

$$\begin{cases} u_i(T_i,S) = u_{i+1}(T_i,S) + \omega_i(S-K_i)^+ \text{ on } (0,\infty) \\ \partial_t u_i + \inf_{\sigma\in[\underline{\sigma},\overline{\sigma}]}\Big[\mathcal{A}_S^{bs}u_i - \lambda\big(u_i - f\sigma S|\partial_S u_i - \eta|\big)^+\Big] - ru_i = 0 \\ \qquad\qquad \text{on } [T_{i-1},T_i)\times(0,\infty). \end{cases} \tag{11}$$

Equations in the Markovian Setting

Let u be defined by $u_i(t,S)$ on each strip $(T_{i-1},T_i]\times(0,\infty)$. The FVA can be defined as $\Theta^{\lambda=0}-\Theta$ and the ensuing KVA process is given as:

$$\mathrm{KVA}_t(\varrho) = h \operatorname*{ess\,sup}_{\mathbb{Q}'\in\mathcal{Q}(t,\mathbb{Q},\mathbb{F}_+)} \mathbb{E}_t^{\mathbb{Q}'}\Big[\int_t^T e^{-(r+h)(s-t)} f\sqrt{\frac{d\langle\varrho\rangle}{ds}}ds\Big],\quad \mathbb{Q}\ a.s., \tag{12}$$

where $\sqrt{\frac{d\langle\varrho\rangle}{dt}} = a_t^{1/2}S_t\big|\partial_S u(t,S_t)-\eta(t,S_t)\big|$.

In the case where $\eta=(1-\alpha)\partial_S u$ we obtain

$$\mathrm{KVA}_t(\varrho) = w(t,S_t),$$

where

$$\begin{cases} w(T,S)=0 \text{ on } (0,\infty) \\ \partial_t w + \sup_{\sigma\in[\underline{\sigma},\overline{\sigma}]}\big[\mathcal{A}_S^{bs}w + \alpha h f\sigma S|\partial_S u|\big] - (r+h)w = 0 \text{ on } [0,T)\times(0,\infty), \end{cases} \tag{13}$$

Numerical example

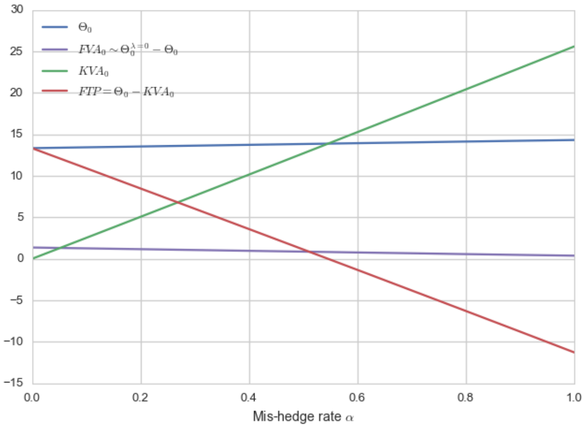

Figure : XVAs and FTP as a function of the mis-hedge parameter α: With UVM uncertainty ($\underline{\sigma} = 15\%, \overline{\sigma} = 60\%$).

Extension and future work

Extension to the setup of Albanese et al. '16 where each option payoff $(S_{T_i} - K_i)^+$ is replaced by the CVA exposure of the bank to the default of its counterparty i, at the (random) time T_i, with corresponding position of the bank $\omega_i S_{T_i}$ and margins received by the bank $\omega_i K_i$.

In this case a relevant risk measure really needs to be computed at a one-year horizon (not instantaneous anymore), in order to leave time to credit events to develop.

Thank you for your attention!