Asymptotic Expansion Pricing of Discretely Monitored Barrier Options under Stochastic Volatilities with Jumps on Returns

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Outline

- Literature review
- Recursion structure
- Expansion: correcting Black-Scholes model
- Numerical experiments

Discretely monitored barrier options

- The pricing of European options has been well studied during the past decades, while the pricing of path-dependent options remains formidable.

- As one class of the most popular path-dependent options, discretely monitored barrier options have the payoff function like

$$(K - S(T))^{+} 1_{\{S(t_1)>B,\, S(t_2)>B,\cdots,S(t_n)>B\}},$$

  where $\{S_t\}$ indicates the price process of the underlying asset, and B is the barrier level.

- Other than true options traded in derivatives markets, the discretely monitoring and barrier structure are often applied in the modeling of credit risk default events.

Literature review

- Broadie, Glasserman and Kou (1997) adjusts the barrier from continuous monitored barrier options
  - Elegant and practical, but limited to the BSM.
- Gaussian models
  - Eydeland (1994) FFT
  - Broadie and Yamamoto (2005) fast Gauss transform
- Lévy models:
  - Petrella and Kou (2004) Spitzer's identity
  - Feng and Linetsky (2008) fast Hilbert transform
  - Kwok and Zeng (2014) time-changed Levy processes

Feng and Linetsky's approach

Feng and Linetsky's method is based on the recursion formula of the return process $\{X_t\}$:

$$v^{(j-1)}(x) = 1_{\{B,+\infty\}}(x) P_\Delta v^{\{j\}}(x)$$

where Δ is the monitoring interval, and $P_t f(x) := E_x[f(X_t)]$. Then based on the property of Hilbert transform

$$\mathcal{F}(\mathrm{sgn} \cdot g) = i\mathcal{H}[\hat{g}],$$

they derived the recursion formula on the frequency domain:

$$\hat{v}^{(j-1)}(\xi) = \frac{1}{2}\phi_\Delta(-\xi)\hat{v}^{(j)}(\xi) + \frac{i}{2}e^{i\xi B}\mathcal{H}[e^{-i\eta B}\phi_\Delta(-\eta)\hat{v}^{(j)}(\eta)](\xi),$$

where ϕ_t is the characteristic function of X_t.

Our contribution

- We price discrete barrier options under general multidimensional diffusion models with jumps in returns.
- An automatic expansion procedure has been designed to output the general expansion formula for any arbitrary order in principle.
- Our algorithm only involves single inversion, and is thus efficient.
- Theoretical convergence of our expansion can be rigorously validated via the theory of Malliavin calculus.

Recursion formula

- Consider the price of a down-and-out put barrier option

$$\mathbb{E}[e^{-rT}(K - S(T))^{+}1_{\{S(t_1)>B,\, S(t_2)>B,\cdots,S(t_n)>B\}}].$$

Here, we have $t_{i\Delta} = i\Delta$ for $i = 0, 1, 2, 3, ..., n$ with $T = n\Delta$.

- Feng and Linetsky (2008) adopted the following backward induction

$$\begin{aligned}
v^{(n-1)}(S(t_{n-1})) &= e^{-r\Delta t}\mathbb{E}[(K - S(t_n))^{+}1_{\{S(t_n)>B\}}|\mathcal{F}(t_{n-1})], \\
v^{(n-2)}(S(t_{n-2})) &= e^{-r\Delta t}\mathbb{E}[v^{(n-1)}(S(t_{n-1}))1_{\{S(t_{n-1})>B\}}|\mathcal{F}(t_{n-2})], \\
v^{(n-3)}(S(t_{n-3})) &= e^{-r\Delta t}\mathbb{E}[v^{(n-2)}(S(t_{n-2}))1_{\{S(t_{n-2})>B\}}|\mathcal{F}(t_{n-3})], \\
\cdots &= \cdots \\
v^{(0)}(S(0)) &= e^{-r\Delta t}\mathbb{E}[v^{(1)}(S(t_1))1_{\{S(t_1)>B\}}|\mathcal{F}(t_0)].
\end{aligned}$$

Recursion formula

- Feng and Linetsky's method can handle all the exponential Lévy models, since their recursion formula exploits the following identity for the return process $\{X_t : t \geq 0\}$:

$$\mathcal{F}(P_t v^{\{j\}})(\xi) = \phi_t(-\xi)\hat{v}^{\{j\}}(\xi), \quad \text{where } P_t v^{(j)}(x) := E[v^{(j+1)}(X_t)|X_0 = x].$$

- Brief reason:

$$\begin{aligned} P_t v(x) &= \int_{-\infty}^{+\infty} v(s+x) g_x(s+x) ds \\ &= \int_{-\infty}^{+\infty} v(s+x) g_0(s) ds = (\tilde{g}_0 * v)(x), \end{aligned}$$

where g_0 and $\tilde{g}_0$ are the density functions of X_t and $-X_t$, respectively. Taking Fourier transform on both sided yields the desired identity.

“Non-Lévy” challenge

- For those non-Lévy models, there is no longer the spatial homogeneous property $g_x(s+x) \equiv g_0(s)$. This prevent us from developing the whole recursion structure.
- The only overlap of exponential Lévy models and diffusion models is the classical Black-Scholes model. Hence, the pricing problem under diffusion models displays apparent difference from that under Lévy structure.
- To make it work for diffusion models, we have to create a similar convolution structure to make the recursion on transforms go through.

Asymptotic expansion method

- To put it simply, consider a general stochastic volatility model:

$$\begin{aligned} dS(t) &= (r-d)S(t)dt + u\left(X(t)\right)S(t)dW_1(t) + dJ_S(t),\ S(0)=S_{0,} \\ dX(t) &= \mu\left(X(t)\right)dt + \sigma\left(X(t)\right)dW(t),\ X(0)=X_0, \end{aligned}$$

- It nests Heston's stochastic volatility model, CIR model, CEV model, and so on.

- Jumps can be included in the first SDE, which is based on the discussions in Kou, Yu and Zhong (2016).

- Introduce an expansion parameter ϵ, and perturb the SDE:

$$\begin{aligned} dY^{\epsilon}(t) &= \left(r-d-\frac{1}{2}v\left(X^{\epsilon}(t)\right)^2\right)dt + v\left(X^{\epsilon}(t)\right)dW_1(t) + dJ(t),\ Y^{\epsilon}(0)=\log s_0, \\ dX^{\epsilon}(t) &= \epsilon[\mu\left(X^{\epsilon}(t)\right)dt + \sigma\left(X^{\epsilon}(t)\right)dW(t)],\ X^{\epsilon}(0)=x_0. \end{aligned}$$

Asymptotic expansion method

- We mimic Feng and Linetsky's backward induction:

$$v(t_i+,x,y,\epsilon) = e^{-r\Delta t}\mathbb{E}[v(t_{i+1}+,X^\epsilon(t_{i+1}),Y^\epsilon(t_{i+1}),\epsilon)1_{\{Y^\epsilon(t_{i+1})>b\}}|X^\epsilon(t_i)=x,Y^\epsilon(t_i)=y].$$

- Pathwise expansion:

$$\begin{aligned} Y^\epsilon(t) &= \sum_{k=0}^{J} Y_k(t)\epsilon^k + O(\epsilon^{J+1}), \\ X^\epsilon(t) &= \sum_{k=0}^{J} X_k(t)\epsilon^k + O(\epsilon^{J+1}). \end{aligned}$$

- Apply the chain rule to derive the expansion of $v(t_i+,x,y)$.

Iterated Stratonovich integrals

- It turns out that our expansion formula for general order can be expressed by a combination of Iterated Stratonovich integrals with different upper limits.

- Define the iterated Stratonovich integral:

$$J_{\mathbf{i}}(t) := \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \circ dW_{i_n}(s_n) \cdots \circ dW_{i_2}(s_2) \circ dW_{i_1}(s_1),$$

for an index $\mathbf{i} = (i_1, \cdots, i_n) \in \{0, 1, 2, \cdots, d, d+1\}^n$. Here, we assume that $W_0(t) := t$ and $W_{d+1}(t) := t$.

Expansion result of $Y(t)$

- Introduce a metric

$$\|\mathbf{i}\| := \sum_{k=1}^{n} [2 \cdot 1_{\{i_k=0\}} + 1_{\{i_k \neq 0\}}]$$

  of the index **i**.

- The k-th expansion term of $X(t)$ and $Y(t)$ can be expressed as

$$X_k(t) = \sum_{\|\mathbf{i}\|=k} C_{\mathbf{i}}(z) J_{\mathbf{i}}(t),$$

$$Y_k(t) = \sum_{\|\mathbf{i}\|=k} C_{\mathbf{i},m+1}(z) J_{(1,\mathbf{i})}(t) - \frac{1}{2} \sum_{\|\mathbf{i}\|=k} C_{\mathbf{i},m+2}(z) J_{(0,\mathbf{i})}(t) + \sum_{\|\mathbf{i}\|=k-1} C_{\mathbf{i},m+3}(z) J_{(0,\mathbf{i})}(t).$$

  Here, the coefficients like $C_{\mathbf{i},m+1}(z)$ are calculated from the model parameters, and are deterministic.

Leading term: Black-Scholes

The induction over the leading terms at all the monitoring points can be handled by employing Laplace transforms:

$$L_{w,0}^{(n)}(s) = \frac{K}{s}(B^{-s} - K^{-s}) - \frac{1}{s-1}(B^{1-s} - K^{1-s}),$$

$$L_{v^{(j)}}(s) = e^{-r\Delta}\frac{1}{v(x)\sqrt{\Delta t}}e^{sc}L_{w,0}^{(j+1)}(s)L_{\varphi_0}(s),$$

$$L_{w,0}^{(j)}(s) = \frac{1}{2}L_{v^{(j)}}(s) + \frac{i}{2}e^{-i\mathrm{Im}(s)b}\mathcal{H}\left[e^{i\mathrm{Im}(s)b}L_{v^{(j)}}(s)\right](-\mathrm{Im}(s)),$$

$$v^{(0)}(x,y,0) = \frac{1}{2\pi}P.V.\int_{-\infty}^{+\infty} e^{s\log(S_0)}L_{v^{(0)}}(s)d\mathrm{Im}(s) \quad \text{for } y > b,$$

where $w(y) := v(y)1_{\{y>b\}}$ and

$$L_{w,0}^{(j)}(s;x) := \int_{-\infty}^{+\infty} e^{-sy}v(t_j+,x,y,0)1_{\{y>b\}}dy, \quad L_{\varphi_0}(s) = v(x)\sqrt{\Delta}\exp\left\{\frac{1}{2}s^2v^2(x)\Delta\right\}.$$

It is exactly the same with the recursion algorithm designed by Feng and Linetsky (2008), when it is restricted to the BSM.

Correction terms

- For the correction terms, the induction can be obtained from a combination among the derivatives of transforms, and the expansion terms like X_k and Y_k.

- Induction on a mesh

- For example, the first correction term can be calculated based on the recursion

$$\frac{\partial}{\partial \epsilon} w^{(j)}(x,y,\epsilon) = e^{-r\Delta t}\mathbb{E}\left(\left[\begin{array}{c} w_1^{(j+1)}(x, Y_0(t_{j+1})) \\ +\frac{\partial}{\partial y} w_0^{(j+1)}(x, Y_0(t_{j+1}))Y_1(t_{j+1}) \\ +\frac{\partial}{\partial x} w_0^{(j+1)}(x, Y_0(t_{j+1}))X_1(t_{j+1}) \end{array}\right] \middle| X^\epsilon(t_j) = x, Y^\epsilon(t_j) = y\right) \cdot 1_{\{y>b\}},$$

- On the frequency domain, the induction can be expressed like

$$L_{w,1}^{(j)}(s) = L_\epsilon^{(j+1)}(s)L_{p_\epsilon}(s) + L_x^{(j+1)}(s)L_{p_x}(s) + L_y^{(j+1)}(s)L_{p_y}(s),$$

and then followed with Feng and Linetsky's Hilbert transform.

Convergence

Enlightened by Watanabe (1987)'s theory, we prove the following main theorem about the theoretical convergence of our expansion approximation in each step of the induction.

Theorem

Assume that $\sigma(s_0) \neq 0$ and the two functions $\mu(\cdot)$ and $\sigma(\cdot)$ have bounded derivatives of all orders. For any $J = 0, 1, 2, \cdots$, the discretely monitored barrier option price at time t_i+ admits the following asymptotic expansion in the sense of classical calculus

$$v(t_i+, x, y) = \sum_{j=0}^{J} \Omega_j(t_{i+1}+, x, y)\epsilon^{\frac{j}{2}} + O\left(\epsilon^{\frac{J+1}{2}}\right).$$

Numerical experiments under CEV

Discretely monitored barrier down-and-out put options under the CEV model with $\beta = -0.5$

T	K	Monthly monitoring AE3	Monthly MC	Monthly Std. err.	Monthly CPU time	Weekly monitoring AE3	Weekly MC	Weekly Std. err.	Weekly CPU time
3/12	90	0.4538	0.4535	0.0015	0.07	0.4192	0.4201	0.0014	0.3
	95	1.4142	1.4146	0.0030		1.3556	1.3560	0.0029	
	100	3.2287	3.2307	0.0048		3.1457	3.1449	0.0047	
	105	6.0095	6.0119	0.0065		5.9021	5.9027	0.0064	
	110	9.6514	9.6541	0.0079		9.5197	9.5239	0.0078	
6/12	90	0.5493	0.5502	0.0017	0.14	0.4561	0.4550	0.0015	0.6
	95	1.5027	1.5049	0.0033		1.3302	1.3273	0.0030	
	100	3.0909	3.0954	0.0050		2.8328	2.8288	0.0048	
	105	5.3596	5.3656	0.0069		5.0136	5.0111	0.0066	
	110	8.2716	8.2792	0.0086		7.8369	7.8360	0.0084	
1	90	0.3888	0.3874	0.0014	0.3	0.2965	0.2963	0.0012	1.1
	95	1.0434	1.0418	0.0028		0.8567	0.8569	0.0025	
	100	2.1138	2.1112	0.0044		1.8136	1.8138	0.0040	
	105	3.6339	3.6293	0.0062		3.2080	3.2072	0.0058	
	110	5.6015	5.5955	0.0080		5.0425	5.0419	0.0076	

Table: Comparison between our expansion pricing (up to the third order) and the Monte Carlo simulations, under the CEV model with $\beta = -0.5$. Other parameters are $r = 0.03$, $\sigma = 2$, $S_0 = 100$, and $B = 80$. The columns "AE3" represents our asymptotic expansion results up to the third order, and the rows "MC" and "Std. err." report the results produced by Monte Carlo simulations with Euler's discretizations by simulating $1,000,000$ sample paths, and their standard errors, respectively. All the AE3 prices lie in the 95% confidence intervals of the associated MC benchmarks.

Numerical experiments under CEV

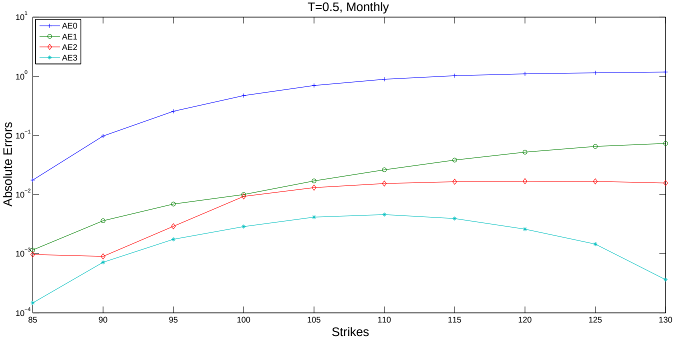
Discretely monitored barrier down-and-out put options under the CEV model with $\beta = -1$

		Monthly monitoring				Weekly monitoring			
T	K	AE3	MC	Std. err.	CPU time	AE3	MC	Std. err.	CPU time
3/12	90	0.4475	0.4477	0.0015	0.07	0.4056	0.4071	0.0014	0.3
	95	1.3723	1.3737	0.0030		1.2998	1.3014	0.0029	
	100	3.1222	3.1261	0.0048		3.0186	3.0195	0.0047	
	105	5.8357	5.8410	0.0065		5.7009	5.7039	0.0064	
	110	9.4360	9.4420	0.0079		9.2699	9.2769	0.0078	
6/12	90	0.5001	0.5003	0.0016	0.14	0.4052	0.4048	0.0014	0.6
	95	1.3748	1.3758	0.0031		1.1950	1.1934	0.0029	
	100	2.8562	2.8593	0.0049		2.5827	2.5805	0.0046	
	105	5.0130	5.0174	0.0067		4.6423	4.6420	0.0064	
	110	7.8315	7.8371	0.0085		7.3622	7.3639	0.0082	
1	90	0.3336	0.3320	0.0013	0.3	0.2481	0.2486	0.0011	1.1
	95	0.9055	0.9028	0.0026		0.7292	0.7303	0.0023	
	100	1.8637	1.8597	0.0042		1.5751	1.5765	0.0038	
	105	3.2594	3.2527	0.0059		2.8435	2.8441	0.0055	
	110	5.1093	5.1004	0.0077		4.5567	4.5572	0.0073	

Table: Comparison between our expansion pricing (up to the third order) and the Monte Carlo simulations, under the CEV model with $\beta = -1$. Other parameters are $r = 0.03$, $\sigma = 20$, $S_0 = 100$, and $B = 80$. The columns “AE3” represents our asymptotic expansion results up to the third order, and the rows “MC” and “Std. err.” report the results produced by Monte Carlo simulations with Euler’s discretizations by simulating $1,000,000$ sample paths, and their standard errors, respectively. All the AE3 prices lie in the 95% confidence intervals of the associated MC benchmarks.

Correction effects

The order-by-order convergence plot under the CEV process



Numerical experiments under Heston jumps

Discretely monitored barrier down-and-out put options under the Heston's model with jumps in returns

T	K	Monthly monitoring AE5	MC	Std. err.	CPU time	Weekly monitoring AE5	MC	Std. err.	CPU time
3/12	90	0.6809	0.6819	0.0027	1.1	0.5455	0.5454	0.0024	5
	95	1.7631	1.7639	0.0050		1.5135	1.5146	0.0046	
	100	3.4826	3.4842	0.0076		3.1105	3.1129	0.0072	
	105	5.8573	5.8606	0.0103		5.3596	5.3638	0.0098	
	110	8.8336	8.8376	0.0128		8.2092	8.2165	0.0123	
6/12	90	0.5106	0.5118	0.0024	2.3	0.3734	0.3727	0.0020	10
	95	1.3144	1.3168	0.0045		1.0415	1.0405	0.0039	
	100	2.5776	2.5806	0.0069		2.1455	2.1438	0.0062	
	105	4.3149	4.3159	0.0095		3.7096	3.7051	0.0088	
	110	6.5035	6.5043	0.0122		5.7176	5.7111	0.0115	
1	90	0.3028	0.3044	0.0018	4.5	0.2144	0.2137	0.0015	20
	95	0.8006	0.8007	0.0035		0.6124	0.6093	0.0030	
	100	1.6012	1.5981	0.0056		1.2852	1.2789	0.0050	
	105	2.7239	2.7182	0.0079		2.2594	2.2488	0.0072	
	110	4.1666	4.1592	0.0105		3.5387	3.5231	0.0096	

Table: Comparison between our expansion pricing (up to the third order) and the Monte Carlo simulations, under the Heston's model with Merton jumps in returns. Other parameters are $r = 0.03$, $\kappa = 2$, $\theta = 0.04$, $\sigma = 0.2$, $\rho = -0.5$, $\mu_J = -0.02$, $\sigma_J = 0.04$, $\lambda = 1.5$, $S_0 = 100$, and $B = 80$. The columns "AE5" represents our asymptotic expansion results up to the fifth order, and the rows "MC" and "Std. err." report the results produced by Monte Carlo simulations with Euler's discretizations by simulating $500,000$ sample paths, and their standard errors, respectively. All the AE5 prices lie in the 95% confidence intervals of the associated MC benchmarks.

Potential future work

- We plan to generate the expansion formula to 6th order practically, and report more accurate prices.
- Seek error bounds.
- Design similar algorithms to approximate the prices of lookback options and Bermudan options.
- More applications on applied probability and risk managements.

Conclusion

- We proposed an expansion pricing method of discretely monitored barrier options under general multidimensional diffusion models with jumps on returns.
- The leading term of the expansion is exactly the pricing formula under the Black-Scholes model provided by Feng and Linetsky (2008), which also serves as the building blocks in our correction terms.
- The convergence of our expansion approximation can be rigorously verified via Malliavin calculus.
- The computational cost of our algorithm is $O(3^J MN)$, which is small enough for fixed J. Current numerical experiments up to the second order demonstrate adequate accuracy.

Thank you!

Q & A