Optimal Labor Supply and Inflation Risk

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Outline

Introduction

Inflation uncertainty

- Inflation is the increase in the price of goods, services, commodities and/or wages
- Although it is not as dramatic as market crash, inflation can be even more devastating in the long run, steadily eroding the values year after year (inflation risk!)
  - Reduction of purchasing power
  - Disruptions to stock and bond markets
  - Devaluation of income
- For investor, in the long term, inflation erodes a portfolio's purchasing power (permanent effect)
  - If an average inflation rate is 3% per year, the value of a portfolio is cut in half every 23 years or so
  - In this respect, the impact of inflation is every bit as severe as that of a market crash and even more devastating in the long run
- Thus, Hedging inflation risk becomes so important.

Introduction

Motivation

- How does the inflation risk affect the household's economic behavior?
- Since inflation risk becomes more severs in the long run, how does it affect the long-term financial planning such as labor market participation and retirement decision.
- Tools for inflation hedging: inflation-linked bond
  - TIPS(Treasury Inflation-Protected Securities) in US
  - indexed government bonds for England, Canada, Australia, and other countries

Literature Review

Inflation risk

- Fischer (1975), Gong and Li (2006), Campbell and Viceira (2001), Brennan and Xia (2002), Munk et. al (2004), Han and Hung (2012), Kwak and Lim (2014), Han and Hung (2017)

Labor-leisure choice

- Bodie et. al (1992), Bodie et. al (2004), Choi et. al (2008), Chai et. al (2011), Perera(2013), Shim and Shin (2014)

Life-insurance purchases

- Pliska and Ye (2007), Ye (2007), Huang and Milevsky (2008), Kwak et al. (2011), Pirvu and Zhang (2012), Kwak and Lim (2014), and many others

Voluntary retirement

- Karatzas and Wang (2000), Choi and Shim (2006), Farhi and Panageas (2007), Dybvig and Liu (2010), Lim and Shin (2011), and many others

Contributions

Contributions

- First study which considers portfolio selection, life insurance, labor-leisure choice, retirement decision, and inflation risk together.
- Explicit solution is derived and quantitative results are provided.
- We find that the changes of the expected inflation rate and the volatility of inflation rate have significant impact on household's long run financial planning, especially on participation in labor market and retirement decision
  $\Rightarrow$ Taking into account inflation risk is important!

Preference

Preference

- A time-additive Cobb-Douglas utility function of consumption rate and leisure rate is considered:

$$u(c(t), l(t)) = \frac{1}{\delta}\left(\frac{c(t)^{\delta} l(t)^{1-\delta}}{1-\gamma}\right)^{1-\gamma}, \qquad 0 < \delta < 1, \quad 0 < \gamma \neq 1$$

- δ and γ represent the elasticity of consumption and the coefficient of relative risk aversion
- If we set $\gamma_1 \triangleq 1 - \delta(1-\gamma)$, then the utility function is rewritten as

$$u(c(t), l(t)) = \frac{c(t)^{1-\gamma_1} l(t)^{\gamma_1 - \gamma}}{1-\gamma_1}.$$

Market Environment

- Assets in the financial market:
  1. Money Market Account:

$$dB(t)/B(t) = Rdt$$

    - R is a nominal risk-free interest rate
  2. Stock (Market Index):

$$dS(t)/S(t) = \mu_s dt + \sigma_s dZ(t)$$

    - μ_s and σ_s are constant return and volatility
    - Z_t is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$
  3. Inflation-linked Index Bond:

$$\begin{aligned} dI(t)/I(t) &= r(t)dt + dP(t)/P(t) \\ &= (r(t) + \mu_p(t))dt + \sigma_p(\rho dZ(t) + \sqrt{1-\rho^2}dU(t)) \end{aligned}$$

    - $r(t)$ is a real interest rate
    - The price process

$$dP(t)/P(t) = \mu_p(t)dt + \sigma_p(\rho dZ(t) + \sqrt{1-\rho^2}dU(t)),$$

    - $Z(t)$ and $U(t)$ are independent Brownian motions

Market Environment

Inflation Risk Modeling

- Inflation Risk is modeled by Price Level Process $P(t)$ (e.g. Consumer Price Index)
- Nominal Value = Real Value $\times$ $P(t)$, or equivalently,

$$\text{Real Value} = \frac{\text{Nominal Value}}{P(t)}.$$

- In our economy, the expected inflation rate $\mu_p(t)$ and real interest rate $r(t)$ are assumed to be constant $(\mu_p(t) = \mu_p, r(t) = r)$.
- In the presence of inflation risk, nominal values should be adjusted as real terms!

Life Insurance Market

- τ_d: death time of household's breadwinner (with intensity $\lambda(t)$)
- Family pays life insurance premium continuously at nominal rate $p_N(t)$.
- At breadwinner's death time τ_d, the bereaved family receives lump-sum insurance benefit $\frac{p_N(\tau)}{\lambda(\tau)}$.

Nominal Wealth Process

Nominal wealth process

- $c_N(t)$: nominal consumption rate
- $l_N(t)$: nominal leisure rate
- $w_N(t)$: labor wage rate
  - nominal income: $w_N(t)(\bar{L}_N - l_N(t))$, where $\bar{L}_N$ is the maximum leisure
- $p_N(t)$: nominal life insurance premium rate
- τ_r: voluntary retirement time
- $\pi_0(t)$, $\pi_1(t)$, and $\pi_2(t)$: proportion of wealth invested in money market account, index bond, and stock $(\pi_0(t) + \pi_1(t) + \pi_2(t) = 1)$
- $X_N(t)$: nominal value of wealth
- Nominal value of wealth $X_N(t)$ follows:

$$dX_N(t) = X_N(t)\left(\pi_0(t)\frac{dB(t)}{B(t)} + \pi_1(t)\frac{dI(t)}{I(t)} + \pi_2(t)\frac{dS(t)}{S(t)}\right) - c_N(t)dt - p_N(t)dt + w_N(t)(\bar{L}_N - l_N(t))dt, \qquad t < \min(\tau_r, \tau_d). \quad (1)$$

Real wealth process

- $c(t) = c_N(t)/P(t)$: real consumption rate
- $l(t) = l_N(t)/P(t)$: real leisure rate
  - real income steam: $w(t)(\bar{L} - l(t))$
  - $0 < l(t) < L < \bar{L}$, for $t \leq \min(\tau_r, \tau_d)$
- $p(t) = p_N(t)/P(t)$: real life insurance premium rate
- $X(t) = X_N(t)/P(t)$: real value of wealth
- $M(t) = X(t) + p(t)/\lambda(t)$: real value of the total legacy
- We consider the case where $w(t) = w$ and $\lambda(t) = \lambda$
- Real value of wealth $X(t)$ follows

$$\begin{aligned}
dX(t) &= d\left(X_N(t)/P(t)\right) \\
&= X_N(t) \cdot d\left(1/P(t)\right) + 1/P(t) \cdot dX_N(t) + d\left(1/P(t)\right) \cdot dX_N(t) \\
&= rX(t)dt + w(t)(\bar{L} - l(t))\mathbf{1}_{\{t \leq \tau_r \wedge \tau_d\}}dt - c(t)dt - p(t)dt \\
&\quad + \pi_0(t)X(t)\left\{(R - r - \mu_p + \sigma_p^2)dt - \rho\sigma_p dW(t) - \sigma_p\sqrt{1-\rho^2}dZ(t)\right\} \\
&\quad + \pi_2(t)X(t)\left\{(\mu_s - r - \mu_p - \rho\sigma_s\sigma_p + \sigma_p^2)dt + (\sigma_s - \rho\sigma_p)dW(t) - \sigma_p\sqrt{1-\rho^2}dZ(t)\right\}
\end{aligned}$$

Budget Constraint

- Define

$$\theta_1 = \frac{\mu_s - R - \rho\sigma_s\sigma_p}{\sigma_s}, \qquad \theta_2 = \frac{r + \mu_p - R - \sigma_p^2}{\sigma_p\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}}\theta_1$$

and

$$\xi(t) \triangleq \exp\left\{-\theta_1 W(t) - \theta_2 Z(t) - \frac{1}{2}\left(\theta_1^2 + \theta_2^2\right) t\right\}.$$

- By Girsanov theorem, a new measure is given by

$$\mathbb{Q}(A) = \mathbb{E}\left[\xi(T)\mathbf{1}_A\right], \qquad A \in \mathcal{F}(T),$$

under which the following processes are standard Brownian motions,

$$d\widetilde{W}(t) = dW(t) + \theta_1 dt, \qquad d\widetilde{Z}(t) = dZ(t) + \theta_2 dt$$

- Under the new measure $\mathbb{Q}$, the inflation-adjusted real wealth dynamics is rewritten as

$$\begin{aligned} dX(t) = {} & rX(t)dt + w(t)(\bar{L} - l(t))\mathbf{1}_{\{t \le \tau_r \wedge \tau_d\}}dt - c(t)dt - p(t)dt \\ & + \{\pi_2(t)\sigma_s - (\pi_0(t) + \pi_2(t))\rho\sigma_p\} X(t)d\widetilde{W}(t) \\ & - (\pi_0(t) + \pi_2(t))X(t)\sigma_p\sqrt{1-\rho^2}d\widetilde{Z}(t). \end{aligned}$$

Budget Constraint

- Define the pricing kernel

$$H(t) \triangleq e^{-(r+\lambda)t} \cdot \xi(t)$$

- By optional sampling theorem and Fatou's lemma with pricing kernel $H(t)$, the wealth dynamics is converted into the following static budget constraint,

$$\mathbb{E}\left[\int_0^{\tau_r} H(t)c(t)dt + \int_0^{\tau_r} \lambda(t)H(t)M(t)dt + \int_0^{\tau_r} H(t)w(t)l(t)dt + H(\tau_r)X(\tau_r)\right] \leq x + \mathbb{E}\left[\int_0^{\tau_r} H(t)w(t)\bar{L}dt\right]. \tag{2}$$

The Problem

The household's objective is to maximize their expected utility by determining $c(t), l(t), p(t), \pi_0(t), \pi_1(t), \pi_2(t)$, and τ_r optimally:

$$
\begin{aligned}
V(x) &\triangleq \sup_{(c,\pi,p,l,\tau_r)} \mathbb{E}\left[\int_0^{\tau_d} e^{-\beta t} u(c(t), l(t))dt + e^{-\beta \tau_d} \bar{u}(M_{\tau_d})\right] \\
&= \sup_{(c,\pi,p,l,\tau_r)} \mathbb{E}\left[\int_0^{\tau_d \wedge \tau_r} e^{-\beta t} u(c(t), l(t))dt + \mathbf{1}_{\{\tau_r > \tau_d\}} e^{-\beta \tau_d} \bar{u}(M_{\tau_d})\right. \\
&\qquad \left. + \mathbf{1}_{\{\tau_r \le \tau_d\}} \int_{\tau_r}^{\tau_d} e^{-\beta t} u(c_t, \bar{L})dt + \mathbf{1}_{\{\tau_r \le \tau_d\}} e^{-\beta \tau_d} \bar{u}(M_{\tau_d})\right],
\end{aligned}
$$

subject to the budget constraint (2) where

$$
\bar{u}(M(\tau_d)) = \frac{\widetilde{L}^{\gamma_1 - \gamma} M(\tau_d)^{1-\gamma_1}}{K_M^{\gamma_1}(1-\gamma_1)}
$$

The Problem

- Since the mortality risk is independent of financial risk, the value function is given by

$$V(x) = \sup_{(c,\pi,p,l,\tau_r)} \mathbb{E}\left[\int_0^{\tau_r} e^{-(\beta+\lambda)t}\left(\frac{l(t)^{\gamma_1-\gamma}c(t)^{1-\gamma_1}}{1-\gamma_1} + \lambda\frac{\widetilde{L}^{\gamma_1-\gamma}M(t)^{1-\gamma_1}}{K_M^{\gamma_1}(1-\gamma_1)}\right)dt \right.$$
$$\left. + e^{-(\beta+\lambda)\tau_r}\left(\frac{\bar{L}^{\frac{\gamma_1-\gamma}{\gamma_1}} + \lambda\widetilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}/K_M}{\lambda+K_1}\right)^{\gamma_1}\frac{X(\tau_r)^{1-\gamma_1}}{1-\gamma_1}\right],$$

Note that

$$K_1 \triangleq r + \frac{\beta-r}{\gamma_1} + \frac{\gamma_1-1}{2\gamma_1^2}(\theta_1^2+\theta_2^2)$$

Duality Approach

- From the Lagrangian,

$$\mathcal{L} = \mathbb{E}\left[\int_0^{\tau_r} e^{-(\beta+\lambda)t}\left(\frac{l(t)^{\gamma_1-\gamma}c(t)^{1-\gamma_1}}{1-\gamma_1} + \lambda\frac{\widetilde{L}^{\gamma_1-\gamma}M(t)^{1-\gamma_1}}{K_M^{\gamma_1}(1-\gamma_1)}\right)dt \right.$$
$$\left. + e^{-(\beta+\lambda)\tau_r}\left(\frac{\bar{L}^{\frac{\gamma_1-\gamma}{\gamma_1}} + \lambda\widetilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}/K_M}{\lambda+K}\right)^{\gamma_1}\frac{X(\tau_r)^{1-\gamma_1}}{1-\gamma_1}\right]$$
$$- \alpha\cdot\mathbb{E}\left[\int_0^{\tau_r} H(t)c(t)dt + \int_0^{\tau_r}\lambda(t)H(t)M(t)dt \right.$$
$$\left. + \int_0^{\tau_r} H(t)w(t)l(t)dt + H(\tau_r)X(\tau_r)\right]$$
$$+ \alpha\left(x + \mathbb{E}\left[\int_0^{\tau_r} H(t)w(t)\bar{L}dt\right]\right)$$

Duality Approach

- Define the auxiliary function by

$$J(\alpha) = \sup_{(c,\pi,p,l,\tau_r)} \mathbb{E}\left[\int_0^{\tau_r} e^{-(\beta+\lambda)t}\left\{\tilde{u}_1(y(t)) + \lambda\, \tilde{u}_2(y(t))\right\} dt + e^{-(\beta+\lambda)\tau_r}\tilde{u}_3(y(\tau_r))\right],$$

where the conjugate functions $\tilde{u}_1(y), \tilde{u}_2(y)$, and $\tilde{u}_3(y)$ are given by

$$\begin{aligned}
\tilde{u}_1(y(t)) = \mathbf{1}_{\{y\geq\tilde{y}\}}&\left(\frac{\gamma}{1-\gamma_1}\left(\frac{w(1-\gamma_1)}{\gamma_1-\gamma}\right)^{-\frac{\gamma_1-\gamma}{\gamma}} y(t)^{-\frac{1-\gamma}{\gamma}}\right)\\
&+\mathbf{1}_{\{0<y<\tilde{y}\}}\left(\frac{\gamma_1}{1-\gamma_1}L^{\frac{\gamma_1-\gamma}{\gamma_1}} y(t)^{-\frac{1-\gamma_1}{\gamma_1}} - w(t)Ly(t)\right),\\
\tilde{u}_2(y(t)) = \frac{\gamma_1}{K_M(1-\gamma_1)}&\tilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}} y(t)^{-\frac{1-\gamma_1}{\gamma_1}},\\
\tilde{u}_3(y(\tau_r)) = \frac{\gamma_1}{1-\gamma_1}&\left(\frac{\bar{L}^{\frac{\gamma_1-\gamma}{\gamma_1}} + \lambda\tilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}/K_M}{\lambda+K}\right) y(\tau_r)^{-\frac{1-\gamma_1}{\gamma_1}} - \frac{w\bar{L}}{\lambda+r}y(\tau_r),
\end{aligned}$$

and

$$\tilde{y} = \left(\frac{w(1-\gamma_1)}{\gamma_1-\gamma}\right)^{-\gamma_1} L^{-\gamma}.$$

Duality Approach

- Dual variable $y(t)$ is defined by

$$\frac{dy(t)}{y(t)} = (\beta - r)dt - \theta_1 dW(t) - \theta_2 dZ(t)$$

- By F.O.Cs, for $0 < y(t) < \tilde{y}$,

$$c^*(t) = \left(\frac{w(1-\gamma_1)}{\gamma_1 - \gamma}\right)^{-\frac{\gamma_1-\gamma}{\gamma}} y(t)^{-\frac{1}{\gamma}}$$

$$l^*(t) = \left(\frac{w(1-\gamma_1)}{\gamma_1 - \gamma}\right)^{-\frac{\gamma_1}{\gamma}} y(t)^{-\frac{1}{\gamma}}$$

$$M^*(t) = \frac{1}{K_M}\tilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}} y(t)^{-\frac{1}{\gamma_1}},$$

and for $y(t) \geq \tilde{y}$,

$$c^*(t) = L^{\frac{\gamma_1-\gamma}{\gamma_1}} y(t)^{-\frac{1}{\gamma_1}},$$

$$l^*(t) = L,$$

$$M^*(t) = \frac{1}{K_M}\tilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}} y(t)^{-\frac{1}{\gamma_1}}.$$

Optimal Policy

Theorem

The optimal wealth process of the primal problem is determined by

$$X^*(t) = \begin{cases} -n_- D y_1^*(t)^{n_- -1} + \frac{(1-\gamma)}{(\lambda+K)(1-\gamma_1)}\left(\frac{w(1-\gamma_1)}{\gamma_1-\gamma}\right)^{-\frac{\gamma_1-\gamma}{\gamma}} y_1^*(t)^{-\frac{1}{\gamma}} & \\ + \frac{\lambda \tilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}}{(\lambda+K_1)K_M} y_1^*(t)^{-\frac{1}{\gamma_1}} - \frac{w\bar{L}}{\lambda+r}, & y(t) \geq \tilde{y} \\ -n_+ C_1 y_2^*(t)^{n_+ -1} - n_- C_2 y_2^*(t)^{n_- -1} & \\ + \frac{1}{\lambda+K_1}\left(L^{\frac{\gamma_1-\gamma}{\gamma_1}} + \frac{\lambda}{K_M}\tilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}\right) y_2^*(t)^{-\frac{1}{\gamma_1}} + \frac{w(L-\bar{L})}{\lambda+r}, & \bar{y} < y(t) \leq \tilde{y}, \\ \frac{1}{\lambda+K_1}\left(\bar{L}^{\frac{\gamma_1-\gamma}{\gamma_1}} + \frac{\lambda}{K_M}\tilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}\right) y_3^*(t)^{-\frac{1}{\gamma_1}}, & 0 < y(t) \leq \bar{y}, \end{cases}$$

and the optimal consumption rate and leisure rate are given by

$$c^*(t) = \begin{cases} \left(\frac{w(1-\gamma_1)}{\gamma_1-\gamma}\right)^{-\frac{\gamma_1-\gamma}{\gamma}} y_1^*(t)^{-\frac{1}{\gamma}}, & -\frac{w\bar{L}}{\lambda+r} < X(t) \leq \tilde{x} \\ L^{\frac{\gamma_1-\gamma}{\gamma_1}} y_2^*(t)^{-\frac{1}{\gamma_1}}, & \tilde{x} < X(t) \leq \bar{x}, \\ y_3^*(t)^{-\frac{1}{\gamma_1}}, & \bar{x} < X(t), \end{cases}$$

and

$$l^*(t) = \begin{cases} \left(\frac{w(1-\gamma_1)}{\gamma_1-\gamma}\right)^{-\frac{\gamma_1}{\gamma}} y_1^*(t)^{-\frac{1}{\gamma}}, & -\frac{w\bar{L}}{\lambda+r} < X(t) \leq \tilde{x} \\ L, & \tilde{x} < X(t) \leq \bar{x}, \\ \bar{L}, & \bar{x} < X(t), \end{cases}$$

Optimal Policy

Theorem (continued)

The life-insurance purchase is determined by

$$p^*(t) = \begin{cases} \lambda\left(\dfrac{\tilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}}{K_M} y_1^*(t)^{-\frac{1}{\gamma_1}} - X^*(t)\right), & -\frac{w\bar{L}}{\lambda+r} < X(t) \le \tilde{x} \\ \lambda\left(\dfrac{\tilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}}{K_M} y_2^*(t)^{-\frac{1}{\gamma_1}} - X^*(t)\right), & \tilde{x} < X(t) \le \bar{x}, \\ \lambda\left(\dfrac{\bar{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}}{K_M} y_3^*(t)^{-\frac{1}{\gamma_1}} - X^*(t)\right), & \bar{x} < X(t), \end{cases}$$

and for $i = 1, 2, 3$, the optimal investment is also given by

$$\pi_2^*(t) = -\frac{1}{\sigma_s\sqrt{1-\rho^2}}\left(\sqrt{1-\rho^2}\theta_1 - \rho\theta_2\right)(y_i^*(t)v''(y_i^*(t))),$$

$$\pi_1^*(t) = X^*(t) - \frac{\theta_2}{\sigma_p\sqrt{1-\rho^2}} y_i^*(t)v_i''(y_i^*(t))$$

$$\pi_0^*(t) = \left(\frac{\theta_2}{\sigma_p\sqrt{1-\rho^2}} + \frac{1}{\sigma_s\sqrt{1-\rho^2}}(\sqrt{1-\rho^2}\theta_1 - \rho\theta_2)\right) y_i^*(t)v_i''(y_i^*(t))$$

Optimal Policy

Theorem (continued)

where for $i = 1$ (or $-\frac{w\bar{L}}{\lambda+r} < X(t) \leq \tilde{x}$),

$$-y_1^*(t)v_1''(y_1^*(t)) = \frac{1}{\gamma}\left\{\left(X(t) + \frac{w\bar{L}}{\lambda+r}\right) + n_-(1-\gamma+\gamma n_-)Dy_1^*(t)^{n_- - 1} + \left(\frac{1}{\gamma} - \frac{1}{\gamma_1}\right)\frac{\lambda\tilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}}{(\lambda+K_1)\gamma_1 K_M} y_1^*(t)^{-\frac{1}{\gamma_1}}\right\} \triangleq \Pi_1(t)$$

and for $i = 2$ (or $\tilde{x} < X(t) \leq \bar{x}$),

$$-y_2^*(t)v_2''(y_2^*(t)) = \frac{1}{\gamma_1}\left\{\left(X(t) + \frac{w(\bar{L}-L)}{\lambda+r}\right) + n_-(1-\gamma_1+\gamma_1 n_-)C_1 y_2^*(t)^{n_- - 1} + n_+(1-\gamma_1+\gamma_1 n_+)C_2 y_2^*(t)^{n_+ - 1}\right\} \triangleq \Pi_2(t),$$

and for $i = 3$ (or $\bar{x} < X(t)$),

$$-y_3^*(t)v_3''(y_3^*(t)) = \frac{X(t)}{\gamma_1} \triangleq \Pi_3(t).$$

Optimal Portfolio

Recall that

$$\theta_1 = \frac{P_S}{\sigma_s}, \qquad \theta_2 = \frac{P_I}{\sigma_p\sqrt{1-\rho^2}} - \frac{\rho P_S}{\sqrt{1-\rho^2}},$$

where

$$P_S \triangleq \mu_s - R - \rho\sigma_s\sigma_p$$

(inflation-adjusted excess return on stock over money market account)

$$P_I \triangleq \mu_p + r - R - \sigma_p^2$$

(inflation-adjusted excess return on index bond over money market account)

Then, for $i = 1, 2, 3$,

- Demand for index bond $\pi_1^*(t) = X(t) - \frac{P_I}{\sigma_p^2(1-\rho^2)}\Pi_i(t) + \frac{\rho P_S}{\sigma_p(1-\rho^2)}\Pi_i(t)$
- Demand for stock $\pi_2^*(t) = \frac{P_S}{\sigma_s}\left(\frac{1}{\sigma_s} + \frac{\rho^2}{1-\rho^2}\right)\Pi_i(t) - \frac{\rho P_I}{\sigma_s\sigma_p(1-\rho^2)}\Pi_i(t)$

Optimal Consumption and Labor Supply

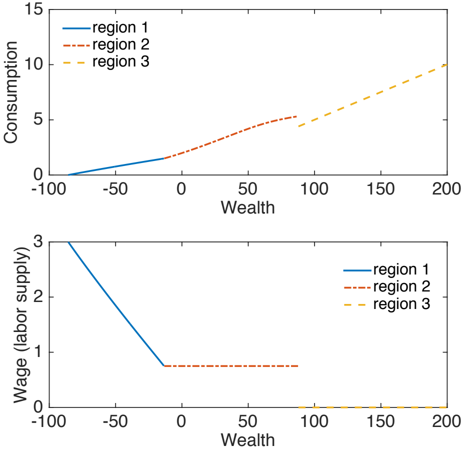

Figure: Optimal consumption rate and wage where the market parameters are given by $\gamma = 3, \delta = 0.4, \beta = 0.08, \mu_s = 0.07, \sigma_s = 0.2, \mu_p = 0.023, \sigma_p = 0.05, R = 0.04, r = 0.02, \rho = -0.07, w = 3, L = 0.75, \bar{L} = 1, \tilde{L} = 0.5.$

Optimal Investment

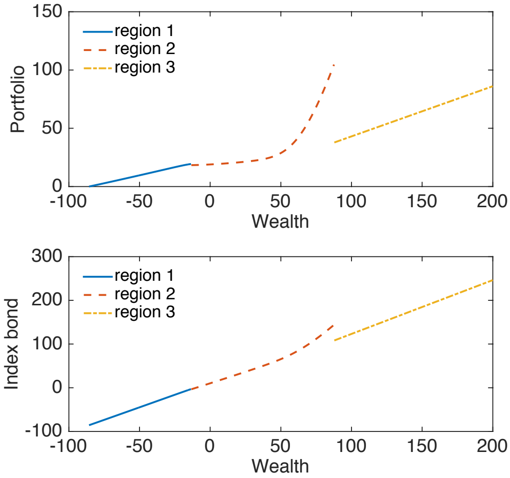

Figure: Optimal investment for risky asset and index bond

Effect of Inflation Rate

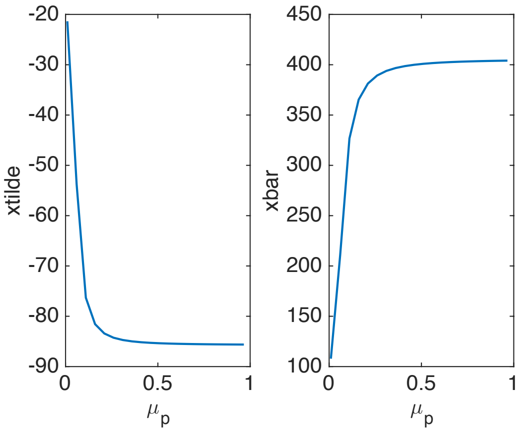

Figure: Effect of expected inflation rate(μ_p) on labor and retirement decision

Effect of Inflation Rate

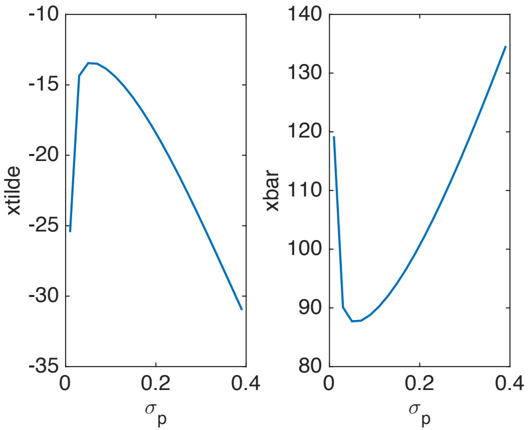

Figure: Effect of fluctuation of inflation rate(σ_p) on labor and retirement decision

Effect of Inflation Rate

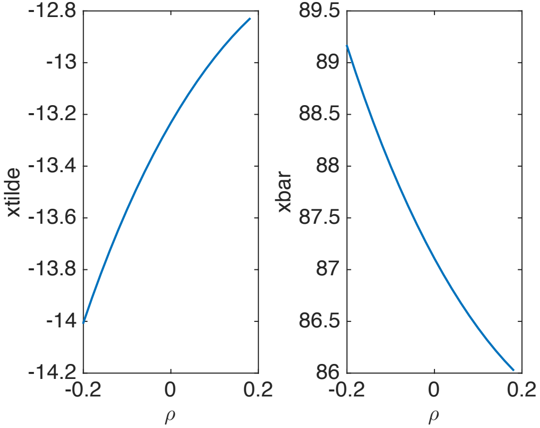

Figure: Effect of correlation between financial market and inflation risk on labor and retirement decision

Concluding Remark

- By duality approach, we find the explicit solutions to the optimal consumption and investment problem are derived when the household faces the inflation uncertainty.
- We provide the quantitative results of optimal policy including consumption, leisure, insurance, investment, and retirement decision.
- The change of inflation rate has a significant impact on household's economic behavior, especially on labor supply.
- The effects of a correlation between financial risk and inflation risk on the labor and retirement decisions are monotone.

Thank you for your attention!

Duality Approach

- The auxiliary function $\phi(t, y)$

$$
\begin{aligned}
&\phi(t,y)\\
&= \sup_{\tau_r} \mathbb{E}\Bigg[\int_t^{\tau_r} e^{-(\beta+\lambda)(s-t)} \Bigg\{ \left(\frac{\gamma}{1-\gamma_1}\right)\left(\frac{w(1-\gamma_1)}{\gamma_1-\gamma}\right)^{-\frac{\gamma_1-\gamma}{\gamma}} y(s)^{-\frac{1-\gamma}{\gamma}} \cdot \mathbf{1}_{\{y \geq \tilde{y}\}} \\
&\qquad + \left(\frac{\gamma_1}{1-\gamma_1} L^{\frac{\gamma_1-\gamma}{\gamma_1}} y(s)^{-\frac{1-\gamma_1}{\gamma_1}} - wLy(s)\right) \cdot \mathbf{1}_{\{0<y<\tilde{y}\}} \\
&\qquad + \frac{\lambda\gamma_1}{K_M(1-\gamma_1)} \widetilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}} y(s)^{-\frac{1-\gamma_1}{\gamma_1}} \Bigg\} ds \\
&\qquad + e^{-(\lambda+\beta)(\tau_r-t)} \left(\frac{\gamma_1}{1-\gamma_1}\right)\left(\frac{\bar{L}^{\frac{\gamma_1-\gamma}{\gamma_1}} + \lambda \widetilde{L}^{\frac{\gamma_1-\gamma}{\gamma}}/K_M}{\lambda + K_1}\right) y(\tau_r)^{-\frac{1-\gamma_1}{\gamma_1}} \Bigg]
\end{aligned}
$$

- The value function is obtained from

$$
V(x) = \inf_{\alpha>0} \left\{ \phi(0,\alpha) + \alpha\left(x + \frac{w\bar{L}}{\lambda + r})\right) \right\}
$$

Duality Approach

Variational inequality

The function $\phi(t,y) \in \mathcal{C}^1((0,\infty)\times\mathbb{R}) \cap \mathcal{C}^2((0,\infty)\times\mathbb{R}/\{\bar{y}\})$ should satisfy the following conditions:

(1) $\mathcal{L}\phi(t,y) + e^{-(\lambda+\beta)t}\left\{\left(\frac{\gamma}{1-\gamma_1}\right)\left(\frac{w(1-\gamma_1)}{\gamma_1-\gamma}\right)^{-\frac{\gamma_1-\gamma}{\gamma}} y^{-\frac{1-\gamma}{\gamma}} + \frac{\lambda\gamma_1 \tilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}}{K_M(1-\gamma_1)} y^{-\frac{1-\gamma_1}{\gamma_1}}\right\} = 0,$

$\tilde{y} \le y$

(2) $\mathcal{L}\phi(t,y) + e^{-(\lambda+\beta)t}\left\{\frac{\gamma_1}{1-\gamma_1}\left(L^{\frac{\gamma_1-\gamma}{\gamma_1}} + \frac{\lambda}{K_M}\tilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}\right) y^{-\frac{1-\gamma_1}{\gamma_1}} - wLy\right\} = 0, \ \bar{y} < y \le \tilde{y}$

(3) $\mathcal{L}\phi(t,y) + e^{-(\lambda+\beta)t}\left\{\frac{\gamma_1}{1-\gamma_1}\left(L^{\frac{\gamma_1-\gamma}{\gamma_1}} + \frac{\lambda}{K_M}\tilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}\right) y^{-\frac{1-\gamma_1}{\gamma_1}} - wLy\right\} \le 0, \ 0 < y < \bar{y}$

(4) $\phi(t,y) = e^{-(\lambda+\beta)t}\left\{\left(\frac{\gamma_1}{1-\gamma_1}\right)\left(\frac{\bar{L}^{\frac{\gamma_1-\gamma}{\gamma_1}} + \lambda\tilde{L}^{\frac{\gamma_1-\gamma}{\gamma}}/K_M}{\lambda+K}\right) y^{-\frac{1-\gamma_1}{\gamma_1}} - \frac{w\bar{L}}{\lambda+r}y\right\}, \ 0 < y < \bar{y}$

(5) $\phi(t,y) \le e^{-(\lambda+\beta)t}\left\{\left(\frac{\gamma_1}{1-\gamma_1}\right)\left(\frac{\bar{L}^{\frac{\gamma_1-\gamma}{\gamma_1}} + \lambda\tilde{L}^{\frac{\gamma_1-\gamma}{\gamma}}/K_M}{\lambda+K}\right) y^{-\frac{1-\gamma_1}{\gamma_1}} - \frac{w\bar{L}}{\lambda+r}y\right\}, \ \bar{y} \le y,$

where the differential operator is defined by

$$\mathcal{L}\phi = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial y}(\beta - r)y + \frac{1}{2}\frac{\partial^2\phi}{\partial y^2}(\theta_1^2+\theta_2^2)y^2.$$

Duality Approach

- From $\phi(t,y) = e^{-(\lambda+\beta)t}v(y)$, we have

$$
\begin{cases}
-(\lambda+\beta)v(y) + (\beta - r)yv'(y) + \frac{1}{2}(\theta_1^2+\theta_2^2)y^2v''(y) & \\
\quad + \left(\frac{\gamma}{1-\gamma_1}\right)\left(\frac{w(1-\gamma_1)}{\gamma_1-\gamma}\right)^{-\frac{\gamma_1-\gamma}{\gamma}} y^{-\frac{1-\gamma}{\gamma}} + \frac{\lambda\gamma_1 \tilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}}{K_M(1-\gamma_1)} y^{-\frac{1-\gamma_1}{\gamma_1}} = 0, & \bar{y} < y \le \tilde{y}, \\
-(\lambda+\beta)v(y) + (\beta - r)yv'(y) + \frac{1}{2}(\theta_1^2+\theta_2^2)y^2v''(y) & \\
\quad + \frac{\gamma_1}{1-\gamma_1}\left(L^{\frac{\gamma_1-\gamma}{\gamma_1}} + \frac{\lambda}{K_M}\tilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}\right) y^{-\frac{1-\gamma_1}{\gamma_1}} - wLy = 0, & \tilde{y} \le y, \\
\left(\frac{\gamma_1}{1-\gamma_1}\right)\left(\frac{\bar{L}^{\frac{\gamma_1-\gamma}{\gamma_1}} + \lambda\tilde{L}^{\frac{\gamma_1-\gamma}{\gamma}}/K_M}{\lambda+K}\right) y^{-\frac{1-\gamma_1}{\gamma_1}} - \frac{w\bar{L}}{\lambda+r}y, & 0 < y < \bar{y}
\end{cases}
$$

Duality Approach

Proposition

The function $v(y)$ is determined from

$$v(y) = \begin{cases} Dy^{n_-} + \frac{\gamma}{(\lambda+K)(1-\gamma_1)}\left(\frac{w(1-\gamma_1)}{\gamma_1-\gamma}\right)^{-\frac{\gamma_1-\gamma}{\gamma}} y^{-\frac{1-\gamma}{\gamma}} \\ \quad + \frac{\lambda\gamma_1 \tilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}}{(\lambda+K_1)K_M(1-\gamma_1)} y^{-\frac{1-\gamma_1}{\gamma_1}}, & \tilde{y} \leq y \\ C_1 y^{n_+} + C_2 y^{n_-} \\ \quad + \frac{\gamma_1}{(\lambda+K_1)(1-\gamma)}\left(L^{\frac{\gamma_1-\gamma}{\gamma_1}} + \frac{\lambda}{K_M}\tilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}\right) y^{-\frac{1-\gamma_1}{\gamma_1}} - \frac{wL}{\lambda+r}y, & \bar{y} < y \leq \tilde{y}, \\ \frac{\gamma_1}{(\lambda+K_1)(1-\gamma_1)}\left(\bar{L}^{\frac{\gamma_1-\gamma}{\gamma_1}} + \frac{\lambda}{K_M}\tilde{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}\right) y^{-\frac{1-\gamma_1}{\gamma_1}} - \frac{w\bar{L}}{\lambda+r}, & 0 < y < \bar{y}, \end{cases}$$

where the coefficients are given by

$$C_2 = \frac{\left(\frac{w(1-\gamma_1)}{\gamma_1-\gamma}\right)^{1-\gamma_1+\gamma_1 n_+} L^{1-\gamma+\gamma n_+}}{(n_+ - n_-)(1-\gamma_1)}\left(\frac{1-\gamma_1+\gamma_1 n_-}{\lambda+K_1} - \frac{1-\gamma+\gamma n_-}{\lambda+K} - \frac{(n_- - 1)(\gamma_1-\gamma)}{\lambda+r}\right),$$

$$C_1 = -\frac{(1-\gamma_1+\gamma n_+)\left(L^{\frac{\gamma_1-\gamma}{\gamma_1}} - \bar{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}\right)}{(n_+ - n_-)(1-\gamma_1)(\lambda+K_1)}\bar{y}^{1-\frac{1}{\gamma_1}-n_-} + \frac{w(L-\bar{L})(n_+ - 1)}{(n_+ - n_-)(\lambda+r)}\bar{y}^{1-n_-},$$

$$D = C_1 + \frac{\left(\frac{w(1-\gamma_1)}{\gamma_1-\gamma}\right)^{1-\gamma_1+\gamma_1 n_-} L^{1-\gamma+\gamma n_-}}{(n_+ - n_-)(1-\gamma_1)}\left(\frac{1-\gamma_1+\gamma_1 n_+}{\lambda+K_1} - \frac{1-\gamma+\gamma n_+}{\lambda+K} - \frac{(n_+ - 1)(\gamma_1-\gamma)}{\lambda+r}\right)$$

Duality Approach

Proposition(continued)

where the free boundary $\bar{y}$ satisfies the following algebraic equation

$$\frac{(1-\gamma_1+\gamma_1 n_-)\left(L^{\frac{\gamma_1-\gamma}{\gamma_1}}-\bar{L}^{\frac{\gamma_1-\gamma}{\gamma_1}}\right)}{(n_+-n_-)(1-\gamma_1)(\lambda+K_1)}\bar{y}^{1-\frac{1}{\gamma_1}-n_+}-\frac{w(L-\bar{L})(n_--1)}{(n_+-n_-)(\lambda+r)}\bar{y}^{1-n_+}$$
$$=\frac{\left(\frac{w(1-\gamma_1)}{\gamma_1-\gamma}\right)^{1-\gamma_1+\gamma_1 n_+}L^{1-\gamma+\gamma n_+}}{(n_+-n_-)(1-\gamma_1)}\left(\frac{1-\gamma_1+\gamma_1 n_-}{\lambda+K_1}-\frac{1-\gamma+\gamma n_-}{\lambda+K}-\frac{(n_--1)(\gamma_1-\gamma)}{\lambda+r}\right)$$

Note that

- K is defined by
$$K=r+\frac{\beta-r}{\gamma}+\frac{\gamma-1}{2\gamma^2}\left(\theta_1^2+\theta_2^2\right)$$
- $n_+(>0)$ and $n_-(<0)$ are two real roots of the quadratic equation
$$\frac{1}{2}\left(\theta_1^2+\theta_2^2\right)n^2+\left(\beta-r-\frac{1}{2}\left(\theta_1^2+\theta_2^2\right)\right)n-(\lambda+\beta)=0$$