Optimal Consumption and Investment under Transaction Costs

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An overview

We revisit the classical Merton consumption and investment problem with proportional transaction costs via a primal dynamic programming approach:

- A transformation scheme to facilitate the analysis of the HJB equation
- Recover some known results in the literature, and offer some new ones
- Potential extension

The Merton problem with proportional transaction costs

The agent can invest in two assets in the economy

1. Risky asset
   - Price process given by $Y = (Y_t)_{t\geq 0}$ an exponential Brownian motion with drift μ and volatility σ:

$$Y_t = Y_0 \exp\left((\mu - \sigma^2/2)t + \sigma B_t\right)$$

   - Purchase and sale of the asset incur proportional transaction cost $\lambda \in [0, \infty)$ and $\gamma \in [0, 1)$ respectively
   - Let Θ_t be the number of units of risky asset held at time t. Write:

$$\Theta_t = \Theta_0 + \Phi_t - \Psi_t$$

     where Φ_t and Ψ_t represent the cumulative units of risky asset purchased and sold respectively

2. Riskfree cash instrument with zero interest rate
   - The agent consumes his cash wealth at a non-negative rate C_t and thus the dynamics his cash wealth $X = (X_t)_{t\geq 0}$ is:

$$dX_t = -C_t dt - Y_t(1+\lambda)d\Phi_t + Y_t(1-\gamma)d\Psi_t$$

The optimal consumption & investment problem

- The CRRA agent with risk aversion coefficient $R \in (0, \infty) \setminus \{1\}$ solves:

$$V(x, y, \theta) = \sup_{(C,\Theta) \in \mathcal{A}(x,y,\theta)} \mathbb{E}\left[\int_0^\infty e^{-\beta t} \frac{C_t^{1-R}}{1-R} dt\right]$$

  - $\beta > 0$ is the agent's subjective discount rate
  - $\mathcal{A}(x, y, \theta)$ is the set of solvent investment/consumption strategies such that

$$X_t + (1-\gamma)\Theta_t^+ Y_t - (1+\lambda)\Theta_t^- Y_t > 0$$

    for all t with initial value $(X_{0-} = x, Y_0 = y, \Theta_{0-} = \theta)$
  - The solvency condition at time t can be rewritten as

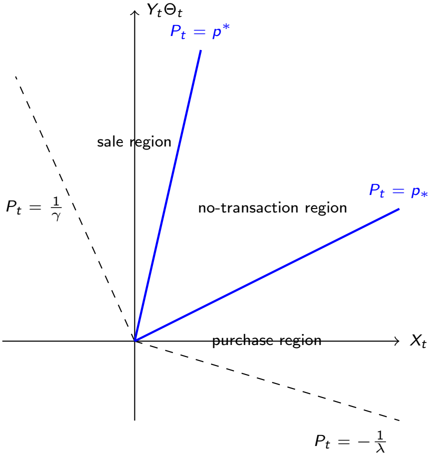
$$-\frac{1}{\lambda} < \frac{Y_t \Theta_t}{X_t + Y_t \Theta_t} < \frac{1}{\gamma}$$

Some background literature

- Conjecture of Constantinides and Magill (1976): trade minimally to keep the fraction of wealth in the risky asset within an interval
- Primal formulation via dynamic programming:
    - Davis and Norman (1990), Shreve and Soner (1994)
- Dual formulation via shadow price:
    - Kallsen and Muhle-Karbe (2010), Choi et al. (2013), Herczegh and Prokaj (2015)
    - Choi et al. (2013) give the most complete treatment of the problem covering all parameter combinations and provide precise conditions for well-posedness

The no-transaction wedge

The conjectured strategy is to keep $P_t := \frac{Y_t\Theta_t}{X_t+Y_t\Theta_t}$ within some no-transaction wedge $[p_*, p^*]$



The free boundary value problem and its transformation

- Write $p := \frac{y\theta}{x+y\theta}$. The candidate value function $V(x,y,\theta)$ is expected to solve:

$$\begin{cases} V_\theta - (1+\lambda)yV_x = 0, & p \in (-1/\lambda, p_*) \\ \frac{R}{1-R} V_x^{-\frac{1-R}{R}} + \mu y V_y + \frac{\sigma^2}{2} y^2 V_{yy} - \beta V = 0, & p \in [p_*, p^*] \\ (1-\gamma)yV_x - V_\theta = 0, & p \in (p^*, 1/\gamma) \end{cases}$$

- Consider the following transformation (away from $p = 1$):
  - $V(x,y,\theta) = \frac{(x+y\theta)^{1-R}}{1-R} \left(\frac{R}{\beta}\right)^R G(p)$
  - $h(p) = \text{sgn}(p(1-p))|1-p|^{R-1} G(p)$
  - $w(h) = p(1-p)h'(p)$ & $W(h) = \frac{w(h)}{(1-R)h}$
  - $q = W(h)$ & $N = W^{-1}$
  - $n(q) = |N(q)|^{-1/R} |1-q|^{1-1/R}$

The first order equation

- Write

$$q_* = \frac{(1+\lambda)p_*}{1+\lambda p_*}, \qquad q^* = \frac{(1-\gamma)p^*}{1-\gamma p^*}$$

which represent the transformed purchase and sale boundary resp

- The transformed equation is

$$\begin{cases} n'(q) = 0, & q \in (-\infty, q_*) \cup (q^*, \infty) \\ n'(q) = O(q, n(q)), & q \in [q_*, q^*] \end{cases}$$

where

$$O(q,n) = -\frac{1-R}{R}\frac{n}{1-q}\frac{n-m(q)}{\ell(q)-n}$$

with

$$m(q) = 1 - \frac{\mu}{\beta}(1-R)q + \frac{\sigma^2}{2\beta}R(1-R)q^2$$

$$\ell(q) = 1 + \left(\frac{\sigma^2}{2\beta} - \frac{\mu}{\beta}\right)(1-R)q - \frac{\sigma^2}{2\beta}(1-R)^2 q^2$$

More on the quadratic function $m(q)$

We will soon see the quadratic function m governs several economically important properties of the problem.

Let (q_M, m_M) be the extrema of m

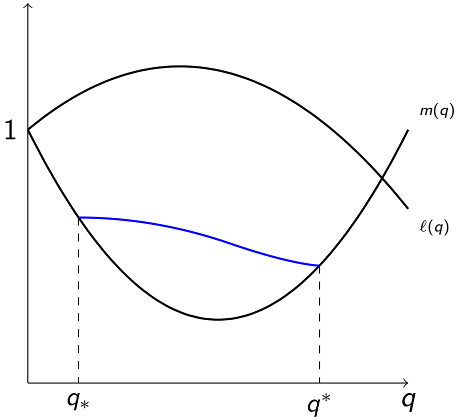
- Minima if $R < 1$; Maxima if $R > 1$
- $q_M = \frac{\mu}{\sigma^2 R}$ is the Merton ratio: the optimal proportion of wealth invested in the risky asset in absence of transaction cost
- $m_M > 0 \iff$ The Merton problem w/o transaction cost is well-posed

Locations of the free boundaries

- The conjectured $C^{1\times 2\times 1}$ smoothness of V translates into C^1 smoothness of n. Then we expect

$$n'(q_*) = n'(q^*) = 0$$

- But then the form of O implies $n(q_*) = m(q_*)$ and $n(q^*) = m(q^*)$
- Looking for a positive solution n to the first order free boundary value problem with end points lying on $m(q)$



An integral constraint

- Define $\xi = \frac{1+\lambda}{1-\gamma} - 1$ such that ξ represents the round-trip transaction cost
- Recall $q_* = \frac{(1+\lambda)p_*}{1+\lambda p_*}$ and $q^* = \frac{(1-\gamma)p^*}{1-\gamma p^*}$. Then

$$1 + \xi = \frac{1+\lambda}{1-\gamma} = \frac{p^*}{1-p^*}\frac{1-p_*}{p_*}\frac{q_*}{1-q_*}\frac{1-q^*}{q^*}$$

and heuristically

$$\ln(1+\xi) = \int_{p_*}^{p^*} \frac{dp}{p(1-p)} - \int_{q_*}^{q^*} \frac{dq}{q(1-q)}$$

- Under the transformation scheme adopted (starting from $w(h) = p(1-p)h'(p)...$), we can deduce

$$\ln(1+\xi) = \int_{q_*}^{q^*} \frac{1}{q(1-q)} \frac{n(q)-m(q)}{\ell(q)-n(q)} dq \tag{1}$$

- The correct choice of q_* and q^* must satisfy (1)

From the free boundary value problem to a family of initial value problems

(For clarity of exposition we will assume $\mu > 0$ and $R < 1$)

Parameterise the family of IVP's by the left starting point u. Define:

- $(n_u(q))_{q \geq u}$ the solution to the IVP $n'(q) = O(q, n(q))$ with $n_u(u) = m(u)$
- $\zeta(u) := \inf(q > u : n_u(q) < m(q))$ the q-coordinate when n_u first crosses m
- $\Sigma(u) := \exp\left(\int_u^{\zeta(u)} \frac{1}{q(1-q)} \frac{n_u(q) - m(q)}{\ell(q) - n_u(q)} dq\right) - 1$

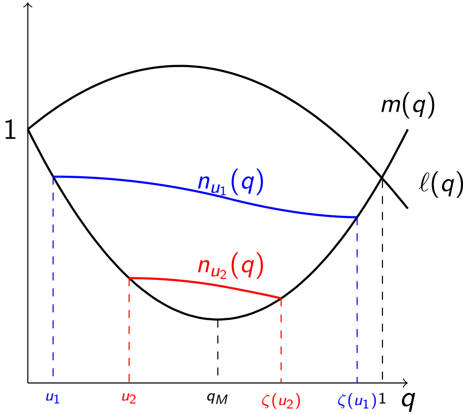
The auxiliary problem

Find a pair $(n_{q_*}(\cdot), q_*)$ such that n_{q_*} is positive and solves $n'(q) = O(q, n(q))$ subject to $n_{q_*}(q_*) = m(q_*)$ and such that $\Sigma(q_*) = \xi$.

An unconditionally well-posed case: $m_M > 0$, $0 < q_M < 1$

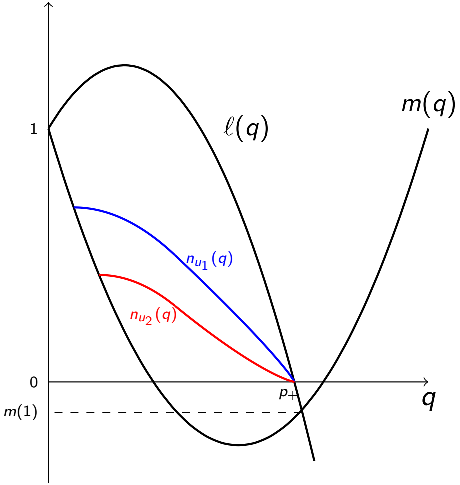
Recall $O(q,n) \propto -\frac{n}{1-q}\frac{n-m(q)}{\ell(q)-n}$:

- For fixed $u \in (0, q_M)$, $n_u(q)$ is decreasing in q until it crosses m again
- $n_u(q)$ strictly bounded above by $\ell(q)$ on $q < 1$
- The family of solutions $(n_u(\cdot))_u$ cannot cross $\implies$ $n_u(\cdot)$ and $\zeta(u)$ are decreasing in u
- Lemma: Σ is a strictly decreasing map from $(0, q_M)$ to $(0, \infty)$
- $q_* = \Sigma^{-1}(\xi)$ and in turn the solution to the auxiliary problem always exist



The ill-posed case: $m_M < 0$ and $m(1) < 0$

- On $q < 1$, all solutions are bounded between 0 and $\ell(q)$
- All solutions must pass through the (singular) point $(p_+, 0)$ where p_+ is the positive root of $\ell(q) = 0$
- No positive solution that starts and ends on m



A conditionally well-posed case: $m_M < 0 < m(1)$, $q_M < 1$

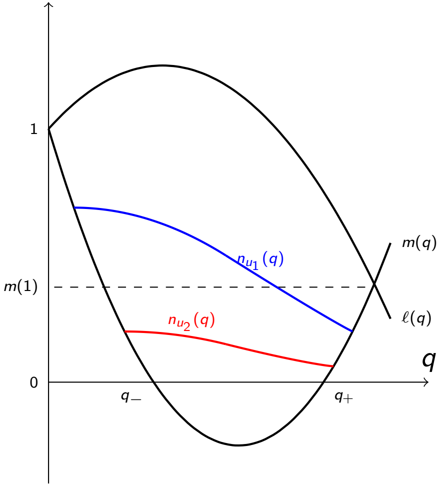
- Write $q_\pm$ the roots of $m(q) = 0$
- Valid left starting point is restricted on $u \in (0, q_-)$
- As $u \uparrow q_-$, $n_u(\cdot) \downarrow 0$ and $\zeta(u) \downarrow q_+$
- Σ should only be defined over $(0, q_-) \to (\underline{\xi}, \infty)$ with

$$\underline{\xi} := \lim_{u \uparrow q_-} \Sigma(u)$$

$$= \exp\left\{ - \int_{q_-}^{q^+} \frac{m(q)}{q(1-q)\ell(q)} dq \right\} - 1$$

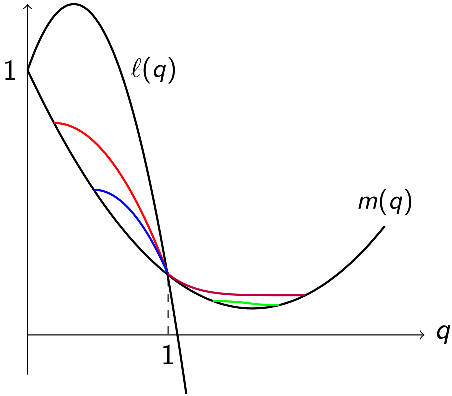
  Can be evaluated analytically

- There does not exist q_* solving $\Sigma(q_*) = \xi$ for $\xi \leqslant \underline{\xi}$



Insensitive right-boundary wrt ξ when $q_M > 1$ and $m_M > 0$

- Any solutions with left starting point $u < 1$ must pass through the (singular) point $(1, m(1))$
- These solutions can be extended uniquely to $q \geq 1$
- For any $u < 1$, $n_u(q) = n_1(q)$ for $q \geq 1$ and $\zeta(u) = \zeta(1)$
- This phenomenon occurs with large transaction cost $\xi > \overline{\xi} := \Sigma(1)$ such that $q_* = \Sigma^{-1}(\xi) < 1$
- Similar behaviour observed by Choi et al. (2013)



A recap of results: well-posedness

Proposition 1

1. If $m_M \geq 0$ then for any $\xi \in (0, \infty)$ there is a unique solution to the auxiliary problem
2. If $m(1) < 0$ then there is no solution to the auxiliary problem for any $\xi \in (0, \infty)$
3. If $m_M < 0$ and $m(1) > 0$ then there is a unique solution to the auxiliary problem if $\xi > \underline{\xi}$, and no solution otherwise

Proposition 2

1. The optimal consumption/investment problem is well-posed iff the auxiliary problem is well-posed (Choi et al. (2013))
2. If $(n_{q_*}(\cdot), q_*)$ solves the auxiliary problem and set $q^* = \zeta(q_*)$, then

$$p_*(\lambda, \gamma) = \frac{q_*(\xi)}{1 + \lambda - \lambda q_*(\xi)}, \qquad p^*(\lambda, \gamma) = \frac{q^*(\xi)}{1 - \gamma + \gamma q^*(\xi)}$$

with $\xi = \frac{\lambda + \gamma}{1 - \gamma}$

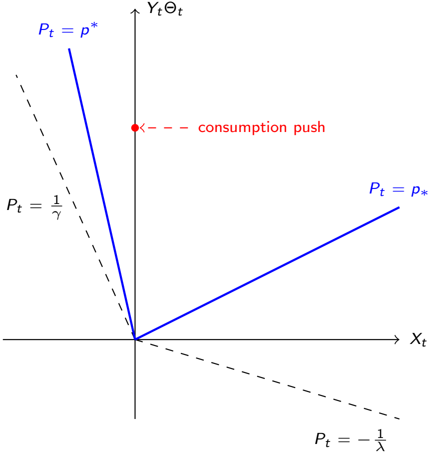
A recap of results: independence of sale boundary on transaction cost

Proposition 3

1. Suppose the auxiliary problem is well-posed. If $q_M > 1$ and $\xi > \overline{\xi}$, then $q^*(\xi) > 1$ and it does not depend on ξ
2. If $\mu > \sigma^2 R$ and $\xi > \overline{\xi}$, then $p^*(\lambda, \gamma) > 1$ and it does not depend on the λ the cost of purchase

A financial interpretation of the insensitive p^* wrt λ

- If the parameters are such that $p^* > 1 > p_*$, the no-transaction wedge inserts the forth-quadrant. The agent may take a leverage position (i.e cash wealth $X_t < 0$)
- Whenever $X_t = 0$, strictly positive consumption will force X_t to be non-positive thereafter
- Hence when the agent sells the risky asset, he will never purchase it again. Any decision regarding sale is irrelevant to the cost of purchase



Some selected results on comparative statics

Proposition 4

Suppose the optimal consumption/investment problem is well-posed

1. If $0 < \mu < \sigma^2 R$:
   - $p*$ (resp. p_*) is increasing (resp. decreasing) in both λ and γ
   - The Merton line lies inside the no-transaction wedge:
     $0 < p_* < q_M = \frac{\mu}{\sigma^2 R} < p^* < 1$

   Otherwise if $\mu \geq \sigma^2 R$, neither of the above needs to hold
2. p_* and p^* are increasing in μ

Very rough ideas of proof:

1. Apply chain rules on the expressions linking p_*^* and q_*^*
2. Comparison of solutions to the auxiliary problem under two different values of μ

Extension

- We can generalise our analysis to an economy with two correlated risky assets - a liquid asset and an illiquid asset
  - Only trading in the illiquid asset incurs a proportional transaction cost
- Same problem studied by Choi (2016) in parallel using shadow price approach under finiteness assumption
- Highlight of results in Hobson, Tse and Zhu (2016a):
  - The key state variable P_t is now the fraction of wealth invested in the illiquid asset
  - Same form of optimal trading strategy: trade the illiquid asset minimally to keep P_t within some no-transaction wedge $[p_*, p^*]$
  - The well-posedness theorem in the one-dimensional case carries over upon re-defining $m(q)$ and $\ell(q)$
  - Some comparative statics results can be derived similarly

Thank you!

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